Difference Equations for Economists[1]

preliminary and incomplete

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March 15, 2021

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Chapter 1

Introduction

1.1 Notation and Preliminaries

Suppose that the state of the economy or some economic system is described in period t by a vector x_t which comprises as its entries the variables of interest. A difference equation or dynamical system then specifies the evolution of this state over *time*. The vector x_t takes value in some normed vector space $\mathbf{X}$ referred to as the *state space*. In almost all cases we identify the state space with $\mathbb{R}^d$, $d = 1, 2, \ldots$, endowed with the Euclidean norm.[1] d is called the dimension of the system. In this monograph the time index t takes on discrete values and typically runs over all integer numbers $\mathbb{Z}$, e.g. $t = \ldots, -2, -1, 0, 1, 2, \ldots$ Sometimes we consider the nonnegative integers $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ only. By interpreting t as the time index, we have automatically introduced the notion of past, present and future.

A difference equation is then nothing but a *rule* or a *function* which instructs how the economic forces transform the current state x_t into next periods state x_{t+1}, *given* current and past states, $x_t, x_{t-1}, \ldots, x_{t-p+1}$, and time t. In its most general form a difference equation can be written as

$$F(x_{t+1}, x_t, x_{t-1}, \ldots, x_{t-p+1}, t) = 0 \tag{1.1}$$

where F is a given function. The difference between the largest and the smallest time index of the state variable explicitly involved is called the *order of the difference equation*. In the formulation (1.1), this is p with $p \geq 1$. In the difference equation above the time index appears explicitly as an argument of the function F. In this case one speaks of a *nonautonomous* or *nonhomogeneous* difference equation. If time is not a separate argument and

[1] See Section C.1 for mathematical definitions and terminology.

enters only as an index of the state variable, the equation is said to be *autonomous* or *homogeneous*. In many applications, the nonautonomous term enters the difference equation by replacing the time index in equation (1.1) by some variable $b_t \in \mathbb{R}^d$. This variable is called the *exogenous* or *independent variable* and may be composed of several variables. F.e. $b_t = Cz_t$ where $z_t \in \mathbb{R}^k$ with $k \geq 1$ and C is a $d \times k$ matrix. The variable x_t is called the *endogenous* or *dependent* variable.

With the exception of Section 3.5, we will always assume that it is possible to solve equation (1.1) uniquely for x_{t+1}:

$$x_{t+1} = f(x_t, x_{t-1}, \ldots, x_{t-p+1}, t) \tag{1.2}$$

The difference equation is called *normal* in this case. Obviously, it is possible to rewrite the above equation as a first order equation by enlarging the state space to become $\mathbb{R}^{dp}$.[2] Thus, in many instances it is sufficient to consider just the first order case:

$$x_{t+1} = f(x_t, t). \tag{1.3}$$

Because $f(., t)$ maps $\mathbf{X}$ into itself, the function f is also called a *transformation*. Most of this monograph considers the following transformations:

linear:	$f(x_t, t) = Ax_t$;
affine:	$f(x_t, t) = Ax_t + b_t$;
linear time-varying:	$f(x_t, t) = A_t x_t$;
affine time-varying:	$f(x_t, t) = A_t x_t + b_t$.

Besides deterministic equations, we will also consider *stochastic difference equations* of the form:

$$A_t x_t = B_t \mathbb{E}_t x_{t+1} - b_t$$

where $\mathbb{E}_t x_{t+1}$ denotes the conditional expectation of x_{t+1} based on information up to period t. In the most general form considered in this monograph the vector of exogenous variables b_t and the matrices A_t and B_t are allowed to vary randomly. Note that the introduction of the conditional expectation induces time to flow in one direction (the natural one) – the difference cannot be reversed. A precise description of the randomness and the conditional expectation is postponed to the corresponding chapters.

Returning to the deterministic case, the system may be initialized at some date t_0 which in most cases is taken to be $t_0 = 0$. Given some starting value x in period 0, the difference equation (1.3) uniquely determines all subsequent

[2] The enlarged state variable then is $(x_t', x_{t-1}', \ldots, x_{t-p+1}')' \in \mathbb{R}^{dp}$.

values of $x_t, t = 1, 2, \ldots$, by iteratively inserting into equation (1.3). In the autonomous case this leads to:

$$\begin{aligned} x_0 &= x \\ x_1 &= f(x_0) = f(x) \\ x_2 &= f(x_1) = f(f(x)) = f \circ f(x) = f^2(x) \\ &\ldots \\ x_t &= f(x_{t-1}) = \underbrace{f \circ \cdots \circ f}_{t \text{ times}}(x) = f^t(x) \end{aligned}$$

where $\circ$ denotes the composition of functions. Hence the value of the state variable in period t is a function of the starting value x. To make this dependence explicit, we write $x_t = \varphi(t, x)$. As a results of this iteration the difference equation generates a sequence $x_t = \varphi(t, x)$, $t = 0, 1, 2, \ldots$ called a *trajectory*. The set of values achieved by a particular trajectory is called an (positive or forward) orbit and is denoted by $\mathcal{O}^+(x) = \{\varphi(t, x) \mid t = 0, 1, 2, \ldots\}$. If f is continuous with a continuous inverse, i.e. if f is a *homeomorphism*, one can iterate the system backward in time to obtain $x_{-1} = f^{-1}(x_0) = f^{-1}(x), x_{-2} = f^{-2}(x_0) = f^{-2}(x), \ldots$ Collecting all values into a set one obtains the (full) orbit $\mathcal{O}(x)$:

$$\begin{aligned} \mathcal{O}(x) &= \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\} \\ &= \{\varphi(t, x) \mid t \in \mathbb{Z}\}. \end{aligned}$$

Figure 1.1 displays the evolution of a dynamic system graphically.

The triplet $(\mathbf{X}, \mathbb{Z}, \varphi)$ defines a *discrete (continuous) dynamical* over time $\mathbb{Z}$, state space $\mathbf{X}$, and a (continuous) function $\varphi : \mathbb{Z} \times \mathbf{X} \to \mathbf{X}$ with the following properties:

(i) $\varphi(0, x) = x$ for all $x \in \mathbf{X}$;

(ii) $\varphi(t + s, x) = \varphi(t, \varphi(s, x))$ for all $t, s \in \mathbb{Z}$ and all $x \in \mathbf{X}$.

These two attributes define the *cocycle properties*. For every $t \in \mathbb{Z}$ the system defines a transformation $\varphi(t, .) = \varphi_t : \mathbf{X} \to \mathbf{X}$. The cocycle properties then read as $\varphi_0 = id_{\mathbf{X}}$ and $\varphi_{t+s} = \varphi_s \circ \varphi_t$. The second property then implies that $\varphi_t \circ \varphi_{-t} = id_{\mathbf{X}}$. Hence, for each $t \in \mathbb{Z}$, φ_t has an inverse given by $\varphi_t^{-1} = \varphi_{-t}$. As a consequence the system is completely determined by φ_1.

Viewed as function of t, $\varphi(., x) = \varphi_x : \mathbb{Z} \to \mathbf{X}$ defines a *solution* to the difference equation. Hence any function $\varphi : \mathbb{Z} \to X$ such that $\varphi(t)$ fulfills the difference equation (1.1), i.e. such that

$$F(\varphi(t + 1), \varphi(t), \varphi(t - 1), \ldots, \varphi(t - p + 1), t) = 0, \tag{1.4}$$

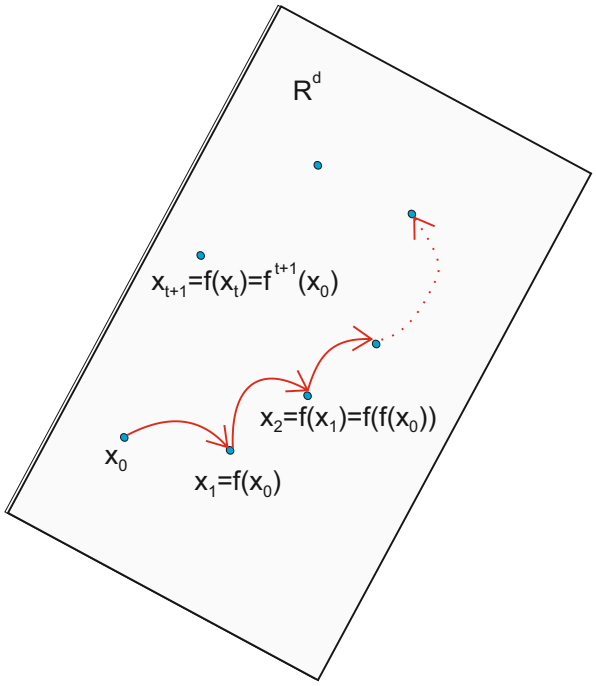


Figure 1.1: A trajectory (orbit) starting in x_0 in period $t_0 = 0$

holds for all $t \in \mathbb{Z}$, is called a solution of the difference equation.

The notation explicitly shows the dependence of the solutions on x. Hence, there usually exists a whole family of solutions indexed by $x \in \mathbf{X}$. To select a unique solution additional requirements such as *boundary conditions* are necessary. The combination of a difference equation with boundary conditions are called *boundary value problems*. In economics these conditions usually take two forms: *initial value conditions* and *terminal conditions*. Initial value conditions are conditions which require that the state variables take a particular value x_{t_0} at some prespecified date t_0. This date is typically the period $t_0 = 0$. If the state variable is a vector, only certain entries may be subject to initial value conditions. Typical economic variables subject to initial value conditions are the capital stock in growth models or the price level in Keynesian models with sticky prices. Such variables are also called *predetermined*. Terminal conditions typically arise in rational expectations models. They determine the initial value x by requiring that $\lim_{t\to\infty} \varphi(t, x)$ remains bounded or converges to some prespecified value, zero for example. If the state variable is a vector, only certain entries may be subject to terminal conditions. Typical economic variables subject to terminal conditions are asset market prices. In many economic models initial value conditions and terminal conditions appear simultaneously as in the Dornbusch model (see Section 4.1) or the optimal growth model (see Section 4.2). If the boundary conditions are sufficient to pin down a unique solution, the economic model is said to be *determinate*. If the boundary conditions are not sufficient to pin down a unique solution such that a whole family of possible solutions remains, the economic model is said to be *indeterminate*.

The aim of the analysis is to assess the existence and uniqueness of a solution to a given difference equation, respectively boundary value problem; and, in the case of many solutions, to characterize the set of all solutions. In addition we are interested in the convergence or divergence of solutions and their sensitivity with respect to the initial value.

1.2 Steady State and Stability

Usually, we are not only interested in describing the evolution of the dependent variable over time, but we also want to know some qualitative properties of the solutions. In particular, we want to characterize its long-run or asymptotic behavior. Consider the nonautonomous first order difference equation (1.3), $x_{t+1} = f(x_t, t)$. Then, an *equilibrium point*, *fixed point* or *steady state* is defined as follows.

Definition 1.1 (Equilibrium Point, Steady State)**.** A point $x^* \in \mathbf{X}$ in the domain of f is called an *equilibrium point*, *fixed point*, or a *steady state* if it satisfies the equation

$$x^* = f(x^*, t), \qquad \text{for all } t \in \mathbb{Z}. \tag{1.5}$$

Equivalently, a solution $\varphi(t, x^*)$ with initial condition $\varphi(0, x^*) = x^*$ satisfies $\varphi(t, x^*) = x^*$ for all $t \in \mathbb{Z}$. Thus, the orbit of x^* just consists of x^*, i.e. $\mathcal{O}(x^*) = \{x^*\}$. It is perfectly possible that the dynamical system possesses more than one equilibrium point.

While the system is at rest once it reached an equilibrium point, the question remains what happens in its vicinity or equivalently what happens when the system is disturbed by some small amount. Will it return to its equilibrium point or will it diverge from it? This issue is treated under the heading of *stability theory*. Using a norm $\|.\|$ on $\mathbf{X}$, the following basic concepts of stability can be defined.[3]

Definition 1.2 (Stability)**.** An equilibrium point x^* is called

- *stable* if for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

  $$\|x_0 - x^*\| < \delta_\varepsilon \quad \text{implies} \quad \|x_t - x^*\| < \varepsilon \quad \text{for all } t > 0, \tag{1.6}$$

  or equivalently

  $$\|x - x^*\| < \delta_\varepsilon \quad \text{implies} \quad \|\varphi(t, x) - x^*\| < \varepsilon \quad \text{for all } t > 0.$$

  Hence, if the system is initiated within a distance δ_ε from the equilibrium point, all subsequent values will remain within a distance ε from the equilibrium point.
  If x^* is not stable, it is called *unstable*.

- *attracting* if there exists $\eta > 0$ such that

  $$\|x_0 - x^*\| < \eta \quad \text{implies} \quad \lim_{t\to\infty} x_t = x^*, \tag{1.7}$$

  or equivalently

  $$\|x - x^*\| < \eta \quad \text{implies} \quad \lim_{t\to\infty} \varphi(t, x) = x^*.$$

  If $\eta = \infty$, x^* is called *globally attracting*. Hence, if the system is initiated within a distance η from the equilibrium point, the sequence $\{x_t = \varphi(t, x)\}$ converges to this equilibrium point.

[3] One obtains similar definitions using a distance instead of a norm.

- *asymptotically stable* or is an *asymptotically stable equilibrium point*[4] if it is stable *and* attracting. If $\eta = \infty$, x^* is called *globally asymptotically stable.*

- *exponentially stable* if there exists $\delta > 0$, $M > 0$, and $\eta \in (0, 1)$ such that for the solution $\varphi(t, x_0)$ we have

$$\|\varphi(t, x_0) - x^*\| \leq M\, \eta^t\, \|\varphi(0, x_0) - x^*\| \text{ whenever } \|x_0 - x^*\| < \delta.$$

- A solution $\varphi(t, x_0)$ is called *bounded* if there exists a positive constant $M < \infty$ such that

$$\|\varphi(t, x_0)\| \leq M, \qquad \text{for all } t.$$

  Thereby the constant M may depend on x_0.

Remark 1.1. Clearly, exponential stability implies stability and, therefore, asymptotic stability and attractiveness. The reverse is, however, not true for each of these implications.

At this point is seems appropriate to analyze some examples.

(i) The simplest example is the linear univariate first order case: $x_{t+1} = \phi x_t$ with $x_t \in \mathbb{R}$ and $\phi \in \mathbb{R} \setminus \{0\}$. Depending on the value of ϕ, there are two possibilities: $\phi \neq 1$ then $x^* = 0$ is the only equilibrium point; $\phi = 1$ then every point in $\mathbb{R}$ is an equilibrium point.
It is easy to see that all solutions are of the form $x_t = \varphi(t, x) = \phi^t x$ (see Section 2.1 in the next chapter). Hence, zero is an exponentially stable equilibrium point if and only if $|\phi| < 1$. If $|\phi| > 1$, $x_t = \varphi(t, x) = \phi^t x$ diverges, except for $x = 0$. Thus, zero is an unstable equilibrium point. If $\phi = -1$, $x_t = \varphi(t, x)$ oscillates between x and $-x$. If $\phi = 1$, $x_t = \varphi(t, x)$ is constant and equal to x for all $t \in \mathbb{Z}$.

(ii) A slight modification to the previous case is achieved when f is an affine instead of a linear function: $x_{t+1} = \phi x_t + b$ where $b \in \mathbb{R} \setminus \{0\}$. Assuming $\phi \neq 1$, the unique equilibrium point is $x^* = b/(1 - \phi)$. For $\phi = 1$ no equilibrium point exists.
Because the difference equation may be written in terms of deviations from steady state, i.e. $(x_{t+1} - x^*) = \phi(x_t - x^*)$, $\phi \neq 1$, the stability analysis for x^* is similar to the previous case. x^* is exponentially stable if $|\phi| < 1$ and unstable if $|\phi| > 1$. If $\phi = -1$, x_t oscillates between x_0 and $-x_0 + b$ implying that x^* is unstable in this case too.

[4] In economics the distinction between stable and attracting is not made. Asymptotic stability is then identified with stability.

(iii) A popular and well-studies example of a nonlinear difference equation is the logistic function with $\mu > 1$:

$$f(x) = \begin{cases} \mu x(1-x), & \text{if } 0 \leq x \leq 1; \\ 0, & \text{otherwise.} \end{cases} \tag{1.8}$$

This function was made popular by May (1976) as a description of population dynamics where the quadratic term is supposed to accommodate saturation effects. The parameter μ thereby captures the surrounding conditions like food supply or climatic conditions. The equilibrium points are determined by the quadratic equation: $x^* = \mu x^*(1-x^*)$. The two solutions are $x^* = 0$ and $x^* = (\mu - 1)/\mu$. Hence the corresponding difference equation has two equilibrium points. $f(x)$ attains a maximum value of $\mu/4$ at $x = 1/2$. The stability analysis is postponed to the end of this Section.

(iv) The following example from Sedaghat (1997) and Elaydi (2005, 181–182) shows that an equilibrium point can be attracting, but unstable:

$$x_{t+1} = f(x_t) = \begin{cases} -2x_t, & \text{for } x_t < \mu; \\ 0, & \text{otherwise,} \end{cases} \tag{1.9}$$

where $\mu > 0$ is a given threshold. It is obvious that $x^* = 0$ is a fixed point. The solutions of this difference equation are

$$x_t = \varphi(t, x) = \begin{cases} (-2)^t x, & \text{if } (-2)^{t-1} x < \mu; \\ 0, & \text{if } (-2)^{t-1} x \geq \mu, \end{cases}$$

where x is some starting value. If $x \geq \mu$, then $x_t = 0$ for all $t \geq 0$. If $x < \mu$, then $f(x_\tau) \geq \mu$ for some $\tau > 0$. Thus, $x_t = 0$ for $t \geq \tau$. The fixed point $x^* = 0$ is therefore attracting, even globally attracting. However, $x^* = 0$ is unstable because points $x \neq 0$, but arbitrarily close to zero, are mapped to points further away until they exceed the threshold μ. Figure 1.3 displays the corresponding Cobweb–diagram.
It can be shown that such a situation can only arise because f is not continuous. In particular, if f is a continuous function on the real line a fixed point cannot be simultaneously attracting and unstable (see Sedaghat, 1997; Elaydi, 2005).

In the univariate case it is usually very convenient to represent the location of equilibrium points and the dynamics graphically. For this purpose, draw first the graph of the function $y = f(x)$ in the (x_t, x_{t+1}) plane. Then, draw the graph of the identity function $y = x$ which is just a line through

the origin having an angle of 45^0 with the x-axis and is denoted by Δ. The equilibrium points are the points where Δ intersects with the graph of the function $y = f(x)$. Starting from some initial value x_0, the evolution of x_t is then represented in the (x_t, x_{t+1})-plane by the following sequence of points. Having computed the first $t \geq 0$ points:

1. Start from $(x_t, 0)$. Move vertically up until you intersect the graph of f at $(x_t, f(x_t))$.

2. Move horizontally until you hit the diagonal Δ at $(f(x_t), f(x_t)) = (x_{t+1}, x_{t+1})$. The projection on the x-axis is $x_{t+1} = f(x_t)$.

3. Move vertically until you hit the graph of f, then again horizontally until you hit the diagonal.

4. Repeat the steps above.

At step $t \geq 1$, one hits the graph of f at $(f^{t-1}(x_0), f^t(x_0))$ and the diagonal Δ at $(f^t(x_0), f^t(x_0))$. Hence, the projection on the x-axis of the points obtained gives the orbit $\mathcal{O}(x_0)$. Connecting these points by line segments gives the so-called *stair step* or *Cobweb*–diagram.

Figure 1.2 shows displays the graphical analysis of the logistic function (1.8) taking $\mu = 2.5$. The two steady states are $x^* = 0$ and $x^* = 0.6$. Obviously, the first steady state is unstable whereas the second one seems to be stable. The solution $\varphi(t, 0.1)$ converges and approaches $x^* = 0.6$ in a circular fashion.

Figure 1.3 displays the Cobweb-diagram for the function (1.9). It shows how the trajectory spirals away from the equilibrium point just to return to it once the threshold is exceeded.

1.3 Characterizing Stability

While the above definitions of stability make intuitively sense, a direct verification of the conditions is often difficult. It is therefore important to have easily applicable criteria instead. We will focus first on the univariate case with $f : \mathbb{R} \to \mathbb{R}$.

1.3.1 The Univariate Case

A useful criterion for asymptotic stability of fixed points in a situation where f is continuous, but not necessarily differentiable is provided by Elaydi (2005, 182 and Appendix C).

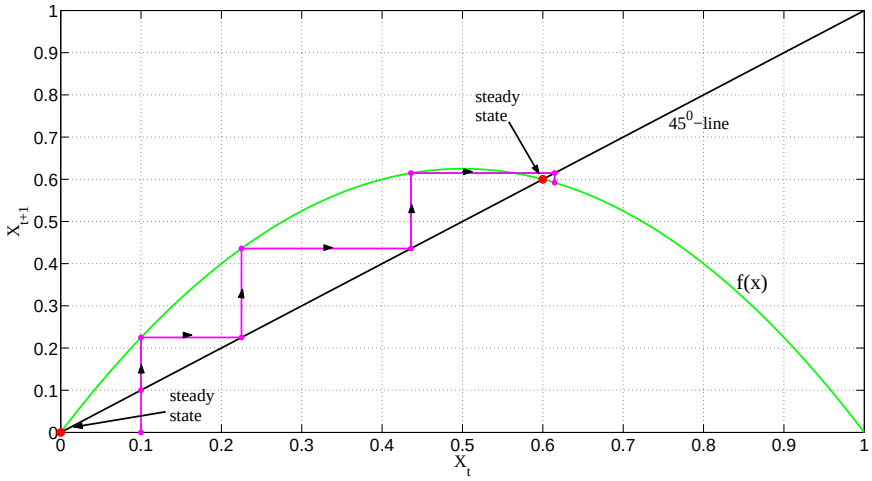


Figure 1.2: Cobweb diagram with steady states of the logistic function: $f(x) = 2.5x(1-x)$ and $x_0 = 0.1$

Theorem 1.1 (Criterion Asymptotic Stability)**.** A fixed point x^* of a continuous function f is asymptotically stable if and only if there exists an open interval (a, b) containing x^* such that $f^2(x) > x$ for $a < x < x^*$ and $f^2(x) < x$ for $x^* < x < b$.

Proof. See Elaydi (2005). □

The most popular criteria are obtained by linearizing the nonlinear equation at the fixed point. This allows to analysis of local stability. In particular, the following theorem holds:

Theorem 1.2 (Stability Condition of Nonlinear Equation)**.** Let x^* be an equilibrium point of the nonlinear autonomous difference equation

$$x_{t+1} = f(x_t)$$

where f is continuously differentiable at x^*. Then,

(i) if $|f'(x^*)| < 1$, then x^* is an asymptotically stable equilibrium point;

(ii) if $|f'(x^*)| > 1$, then x^* is unstable.

Proof. The proof follows Elaydi (2005, 27–28). Suppose that $|f'(x^*)| \leq M$ for some $M < 1$. Then, because of the continuity of the derivative, there

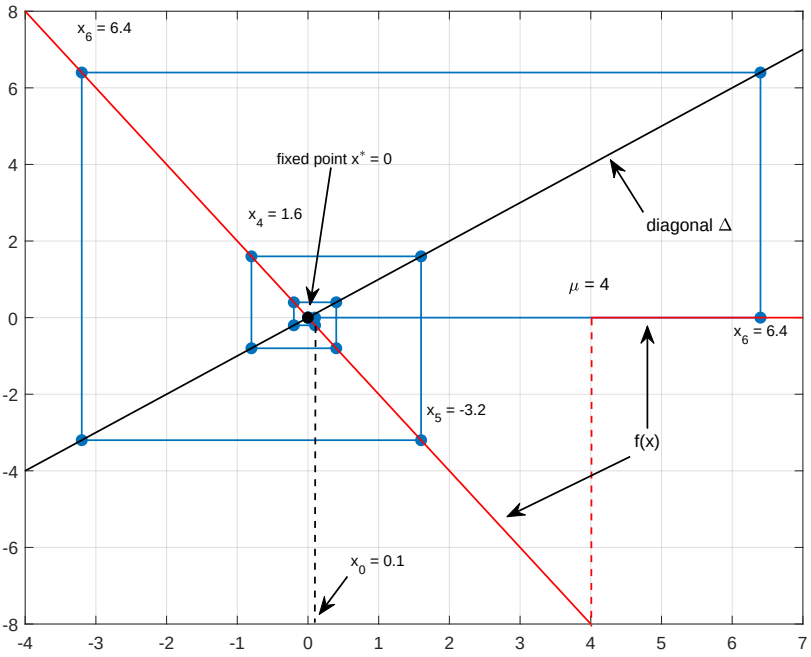


Figure 1.3: Cobweb diagram of function (1.9) with $\mu = 4$ and $x_0 = 0.1$

exists an interval $I = (x^* - \gamma, x^* + \gamma)$, $\gamma > 0$, such that $|f'(x)| \leq M < 1$ for all $x \in I$. For $x_0 \in I$,

$$|x_1 - x^*| = |f(x_0) - f(x^*)|.$$

The mean value theorem A.2 then implies that there exists ξ, $x_0 < \xi < x^*$, such that

$$|f(x_0) - f(x^*)| = |f'(\xi)|\,|x_0 - x^*|.$$

Hence we have

$$|x_1 - x^*| \leq M|x_0 - x^*|.$$

This shows that x_1 is closer to x^* than x_0 and is thus also in I because $M < 1$. By induction we therefore conclude that

$$|x_t - x^*| \leq M^t|x_0 - x^*|.$$

For any $\varepsilon > 0$, let $\delta_\varepsilon = \min\{\gamma, \varepsilon\}$ then $|x_0 - x^*| < \delta_\varepsilon$ implies $|x_t - x^*| < \varepsilon$ for all $t \geq 0$. x^* is therefore a stable equilibrium point. In addition, x^* is attractive because $\lim_{t\to\infty} |X_t - X^*| = 0$. Thus, X^* is asymptotically stable. □

Remark 1.2. The proof shows that the equilibrium is even exponentially stable if $f'(x^*) < 1$.

Remark 1.3. Note that the case $|f'(x^*)| = 1$ is not treated by this theorem. It involves a more detailed analysis which involves higher order derivatives (see Elaydi, 2005, 29–32). In the mathematical literature a fixed point x^* is called a *hyperbolic* fixed point if $|f'(x^*)| \neq 1$. Compare this with the analysis of the second example in Section 1.2 where $f(x) = \phi x + b$.

Example: Newton's method Suppose we want to determine the solution to the equation $g(x) = 0$ and suppose further that there is no analytic solution available so that we must solve the equation numerically. A well-known and popular method is the so-called Newton–Raphson method. Given some guess x_t, the method consists in considering the linearized version $g(x_t) + g'(x_t)(x - x_t) = 0$ of the equation $g(x) = 0$. Solving this equation gives an approximate solution $x = x_t - \frac{g(x_t)}{g'(x_t)}$. Taking this solution as the new starting point x_{t+1} results in a difference equation

$$x_{t+1} = f(x_t) = x_t - \frac{g(x_t)}{g'(x_t)} \tag{1.10}$$

The steady state of this difference equation is then a solution of the original equation, provided $g'(x^*) \neq 0$. The situation is depicted graphically in

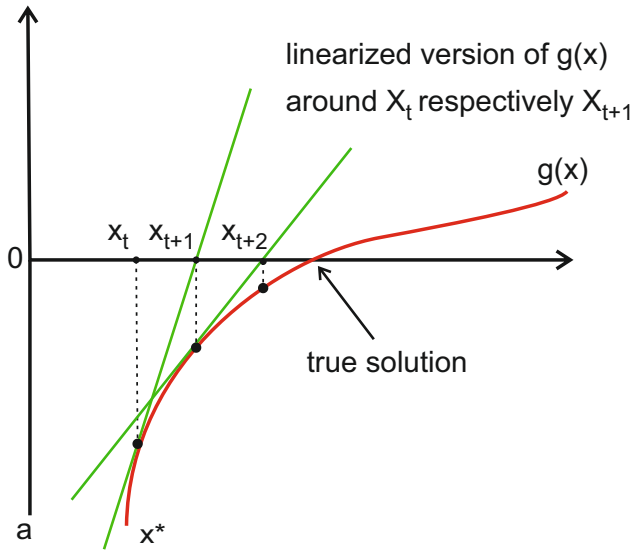


Figure 1.4: The Newton-Raphson method for solving iteratively the equation $g(x) = 0$

Figure 1.4 where the approximate solutions x_t, x_{t+1}, x_{t+2} (the solution to the linearized versions of $g(x) = 0$) approach the true solution.

In order to study the stability of the difference equation, we evaluate the derivative of f at x^*:

$$|f'(x^*)| = \left|1 - \frac{(g'(x^*))^2 - g(x^*)g''(x^*)}{(g'(x^*))^2}\right|.$$

This derivative is zero because $g(x^*) = 0$. Then, by Theorem 1.2, x^* is asymptotically stable. This implies that $\lim_{t\to\infty} x_t = x^*$ provided x_0 is chosen close enough to x^*.

As an illustration consider the computation of the square root of some positive number a. Take for this purpose $g(x) = x^2 - a$. The corresponding difference equation then becomes

$$x_{t+1} = x_t - \frac{x_t^2 - a}{2x_t} = \frac{1}{2}\left(x_t + \frac{a}{x_t}\right).$$

Starting with $x_0 > 0$, the difference equation converges to $\sqrt{a}$ whereas if $x_0 < 0$ the limit is $-\sqrt{a}$. The above difference equation has a nice intuitive interpretation. Suppose $x_t > \sqrt{a}$ then $a/x_t < \sqrt{a}$, thus by taking the arithmetic average between x_t and a/x_t one can expect to get closer to $\sqrt{a}$. The argument holds similarly for $x_t < \sqrt{a}$.

Example: Logistic function The stability of the logistic difference equation at the equilibrium point $x^* = (\mu - 1)/\mu$ depends on the slope of $f'(x^*) = \mu - 2\mu x^* = 2 - \mu$. Thus, according to Theorem 1.2 x^* is asymptotically stable if $1 < \mu < 3$. Starting from any value x in $(0, 1)$ x_t will converge to x^*. For values of $\mu > 3$ complicated dynamics emerge including chaotic behavior. Details can be found in May (1976) and Robinson (1999).

1.3.2 The Multivariate Case

We start by studying the linear autonomous case

$$x_{t+1} = Ax_t \tag{1.11}$$

with $x_t \in \mathbb{R}^d$ and $A \in \mathbb{GL}(d)$.[5] These difference equations have zero as a fixed point and their solutions are given by $x_t = \varphi(t, x) = A^t x$ (see Chapter 3, in particular Section 3.2.1). For these systems the stability is intimately related to the eigenvalues of A, especially to the spectral radius of A, $\rho(A)$.[6] A proof

[5] $\mathbb{GL}(d)$ denotes the general linear group, i.e. the set of all invertible $d \times d$ matrices. A compendium summarizing the most important concepts of linear algebra is provided in Appendix C.

[6] The spectral radius is the largest absolute eigenvalue.

of this assertion is based on the following lemma.

Lemma 1.1. The zero solution of the homogeneous system (1.11) is stable if and only if there exists $M > 0$ such that

$$\left\|A^t\right\| \leq M \quad \text{for all } t \geq 0. \tag{1.12}$$

Proof. Suppose that the inequality (1.12) is satisfied then $\|x_t\| \leq M\|x_0\|$. Thus, for $\varepsilon > 0$, let $\delta_\varepsilon = \frac{\varepsilon}{M}$. Then $\|x\| < \delta_\varepsilon$ implies $\|x_t\| < \varepsilon$ so that the zero point is stable.
Conversely, suppose that the zero point is stable. Then for all $\|x\| < \delta$,

$$\|A^t\| = \sup_{\|\xi\|\leq 1} \|A^t\xi\| = \frac{1}{\delta} \sup_{\|x\|\leq\delta} \|A^t x\| \leq \frac{\varepsilon}{\delta} = M$$

where the first equality is just the definition of the matrix norm (operator norm) corresponding to the norm in $\mathbb{R}^d$. The second equality is a consequence of $\|x\| \leq \delta$. The inequality follows from the assumption that the zero point is a stable equilibrium. □

Remark 1.4. The condition given in equation (1.12) is equivalent to the condition that all solutions are bounded.

Theorem 1.3 (Stability)**.** For the homogeneous linear system in $\mathbf{X} = \mathbb{R}^d$

$$x_{t+1} = Ax_t, \quad A \in \mathbb{GL}(d),$$

the following statements are true:

(i) The zero solution is stable if and only if $\rho(A) \leq 1$ and the eigenvalues on the unit circle are semisimple.

(ii) The zero solution is asymptotically stable if and only if $\rho(A) < 1$. In this case, the solution is even exponentially stable.

Proof. According to the previous Lemma 1.1 we have to prove that $\|A^t\| \leq M$ for some $M > 0$. Using the Jordan canonical form[7] $A = QJQ^{-1}$, this amounts to $\|A^t\| = \|QJ^tQ^{-1}\| \leq M$. But this is equivalent to the existence of a $\widetilde{M} > 0$ such that $\|J^t\| \leq \widetilde{M}$. M may then be taken as $M = \frac{\widetilde{M}}{\|Q\|\|Q^{-1}\|}$. The powers of J are given by powers of the Jordan blocks. The t-th power

[7]For a definition of the Jordan canonical form the reader is referred to Appendix C.

of a Jordan block corresponding to the eigenvalue λ_i is given according to equation (3.12) by

$$\begin{aligned} J_j(\lambda_i)^t &= (\lambda_i I + N)^t \\ &= \lambda_i^t I + \binom{t}{1}\lambda_i^{t-1}N + \binom{t}{2}\lambda_i^{t-2}N^2 + \cdots + \binom{t}{k-1}\lambda_i^{t-k+1}N^{k-1} \\ &= \begin{pmatrix} \lambda_i^t & \binom{t}{1}\lambda_i^{t-1} & \binom{t}{2}\lambda_i^{t-2} & \cdots & \binom{t}{k-1}\lambda_i^{t-k+1} \\ 0 & \lambda_i^t & \binom{t}{1}\lambda_i^{t-1} & \cdots & \binom{t}{k-2}\lambda_i^{t-k+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ & & & & \binom{t}{1}\lambda_i^{t-1} \\ 0 & 0 & 0 & \cdots & \lambda_i^t \end{pmatrix} \end{aligned}$$

where N is a nilpotent matrix of order k, i.e. $N^k = 0$. The elements in this matrix become unbounded if $|\lambda_i| > 1$. They also become unbounded if $|\lambda_i| = 1$ and $J_j(\lambda_i)$ is not a 1×1 matrix. If, however, for all eigenvalues with $|\lambda_i| = 1$ the largest Jordan blocks are 1×1, then the Jordan segment corresponding to λ_i with $|\lambda_i| = 1$, $J(\lambda_i)$, is just a diagonal matrix with ones in the diagonal and is therefore obviously bounded. The Jordan segments, $J(\lambda_i)$, corresponding to eigenvalues $|\lambda_i| < 1$, converge to zero, i.e. $\lim_{t\to\infty} J(\lambda_i)^t = 0$ because $t^k\lambda_i^t \to 0$ as $t \to \infty$ by l'Hôpital's rule. □

Theorem 1.3 showed that the stability properties crucially depend on the location of the eigenvalues relative to the unit circle. If there are no eigenvalues on the unit circle, small perturbations of A will not affect the location of the eigenvalues relative to the unit circle because they depend continuously on the entries of A. Thus, small perturbations preserve the stability properties. This motivates to bring out matrices with no eigenvalues on the unit circle in an own definition.

Definition 1.3 (Hyperbolic Matrix)**.** A *hyperbolic* matrix is a matrix with no eigenvalues on the unit circle. If A is a hyperbolic matrix, then the corresponding linear homogeneous difference equation $x_{t+1} = Ax_t$ is also called hyperbolic.

An immediate implication for hyperbolic matrices is that the zero solution is the unique fixed of the autonomous linear difference equation (1.11).

Definition 1.4 (Hyperbolic Fixed Point)**.** Let x^* be a fixed point of a continuously differentiable map $f : \mathbb{R}^d \to \mathbb{R}^d$, then x^* is called a *hyperbolic fixed point* if and only if the derivative of f at x^*, i.e. the Jacobian matrix $D_{x^*}f$, is invertible and does not have any eigenvalue on the unit circle.

Consider a hyperbolic matrix A with spectrum $\sigma(A) = \{\lambda_1, \ldots, \lambda_s\}$, $1 \leq s \leq d$.[8] Then, partition the spectrum into the stable, respectively unstable eigenvalues, i.e. $\sigma_s(A) = \{\lambda \in \sigma(A) : |\lambda| < 1\}$ and $\sigma_u(A) = \{\lambda \in \sigma(A) : |\lambda| > 1\}$. Denote by $\mathbb{L}^s$ the space generated by the eigenvectors (generalized eigenvectors) corresponding to the eigenvalues in $\sigma_s(A)$ and, similarly, by $\mathbb{L}^u$ the space generated by the eigenvectors (generalized eigenvectors) corresponding to the eigenvalues in $\sigma_u(A)$. The two subspaces span the whole state space such that

$$\mathbb{R}^d = \mathbb{L}^s \oplus \mathbb{L}^u$$

where $\oplus$ denotes the direct sum of two linear spaces. It is worth emphasizing that $\mathbb{L}^s$ and $\mathbb{L}^u$ are not necessarily orthogonal to each other. Given these definitions, we may follow Elaydi (2005) and state the following theorem.

Theorem 1.4 (Stable Manifold Theorem)**.** Given a linear autonomous hyperbolic difference equation

$$x_{t+1} = Ax_t, \quad A \in \mathbb{GL}(d). \tag{1.13}$$

Then the following properties hold:

(i) If $\varphi(t, x)$ is a solution with $x \in \mathbb{L}^s$, then $\varphi(t, x) \in \mathbb{L}^s$ for all t. Moreover,

$$\lim_{t \to \infty} \varphi(t, x) = 0.$$

(ii) If $\varphi(t, x)$ is a solution with $x \in \mathbb{L}^u$, then $\varphi(t, x) \in \mathbb{L}^u$ for all t. Moreover,

$$\lim_{t \to -\infty} \varphi(t, x) = 0.$$

Proof. The proof follows Elaydi (2005, theorem 4.14). Let $\varphi(t, x)$ be a solution with $x \in \mathbb{L}^s$. The definition of the (generalized) eigenvector corresponding to some eigenvalue $\lambda \in \sigma(A)$ implies that $A\mathbb{E}_\lambda = \mathbb{E}_\lambda$ where $\mathbb{E}_\lambda$ denotes the space generated by the (generalized) eigenvectors corresponding to λ. Hence $A\mathbb{L}^s = \mathbb{L}^s$ and $\varphi(t, x) \in \mathbb{L}^s$ for all $t \geq 0$. Given $x \in \mathbb{L}^s$, we can represent x as $x = \sum_{j=1}^{r} \alpha_j e_j$ where r is the number of eigenvalues, distinct or not, strictly smaller than one and e_j denotes the (generalized) eigenvectors corresponding to those eigenvalues. Let $J = Q^{-1}AQ$ be the Jordan form of A. Next, partition the Jordan matrix into blocks of stable, respectively unstable eigenvalues accordingly:

$$J = \begin{pmatrix} J_s & 0 \\ 0 & J_u \end{pmatrix}$$

[8] The spectrum of a matrix is given by the set of its distinct eigenvalues.

where J_s and J_u are $r \times r$, respectively $(n-r) \times (n-r)$ matrices. Therefore,

$$\begin{aligned}\varphi(t,x) = A^t x = QJ^tQ^{-1}x = QJ^tQ^{-1}\sum_{j=1}^{r}\alpha_j e_j \\ = Q\begin{pmatrix} J_s^t & 0 \\ 0 & J_u^t \end{pmatrix}\sum_{j=1}^{r}\alpha_j Q^{-1}e_j = Q\sum_{j=1}^{r}\alpha_j \begin{pmatrix} J_s^t & 0 \\ 0 & J_u^t \end{pmatrix} Q^{-1}e_j \\ = Q\sum_{j=1}^{r}\alpha_j \begin{pmatrix} J_s^t & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}e_j.\end{aligned}$$

The last equality follows from the fact that the $Q^{-1}e_j$'s are of the form $(\xi_{1j}, \ldots, \xi_{rj}, 0, \ldots, 0)'$ which is a consequence of the e_j's being (generalized) eigenvectors. This implies that $\varphi(t,x) \to 0$ as $t \to \infty$ because $J_s^t \to 0$ as $t \to \infty$.

The proof of (ii) is analogous. □

Remark 1.5. Theorem 1.4 may be refined by relaxing the assumption of an hyperbolic matrix and can be sharpened with respect to its conclusions (Colonius and Kliemann, 2014, theorem 1.5.8).

An interesting special case, often encountered in economics, is obtained when the system expands in some directions, but contracts in others. In such a case the fixed point is called saddle point.

Definition 1.5 (Saddle Point)**.** The zero solution of the hyperbolic linear difference equation $x_{t+1} = Ax_t$ is called a *saddle point* if there exist at least two eigenvalues of A, λ_u and λ_s, such that $|\lambda_u| > 1$ and $|\lambda_s| < 1$.

Linearization

The analogy to one dimensional difference equations suggests to analyze the stability properties of nonlinear multivariate systems via a linear approximation around the steady state (compare Theorem 1.2). Consider for this purpose the nonlinear homogeneous system $x_{t+1} = f(x_t)$ with $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and fixed point x^*. Suppose that f is continuously differentiable in an open neighborhood of x^*, then the *linearized difference equation* is given by

$$X_{t+1} - X^* = A(X_t - X^*)$$

where A is the derivative of f evaluated at x^*, i.e. $A = \mathrm{D}_{x^*}f$, the Jacobian matrix of partial derivatives. We say that A is a *linearization* of f if and

only if f is conjugate to A, i.e. if there exists a homeomorphism h such that $A \circ h = h \circ f$. Thus, h makes the diagram below commutative.

$$\begin{array}{ccc} \mathbb{R}^d & \xrightarrow{f} & \mathbb{R}^d \\ {\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow \\ V_1 \subset \mathbb{R}^d & \xrightarrow{A=\mathrm{D}_{x^*}f} & V_2 \subset \mathbb{R}^d \end{array}$$

where V_1 and V_2 are open subsets of $\mathbb{R}^d$. If A is a linearization of f, we can infer the stability properties of f from those of A. Because the latter are well understood, we even have explicit solution formulas at hand (see Section 3.2.2), linearization represents a powerful tool in this respect.

Then, naturally, the question arises: which systems have a linearization and is a system close to a linear one conjugate to it? These questions are answered, at least locally, by the famous Hartman–Grobman theorem (see f.e. Elaydi (2005), Robinson (1999, section 5.6) or Coudène (2016, chapter 8)). This theorem can be seen as an extension of the Stable Manifold Theorem 1.4 to nonlinear difference equations.

Theorem 1.5 (Hartman–Grobman Theorem)**.** If f is a continuously differentiable with hyperbolic fixed point x^*, then there exist neighborhoods $U_1, U_2 \subset \mathbb{R}^d$ of x^* and $V_1, V_2 \subset \mathbb{R}^d$ of 0 and a homeomorphism $h : U_1 \cup U_2 \to V_1 \cup V_2$ that conjugates f locally to $A = D_{x^*}f$, i.e. for all $x \in U_1$ $h(f(x)) = D_{x^*}f(h(x))$ or

$$\begin{array}{ccc} U_1 \subset \mathbb{R}^d & \xrightarrow{f} & U_2 \subset \mathbb{R}^d \\ {\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow \\ V_1 \subset \mathbb{R}^d & \xrightarrow{A=D_{x^*}f} & V_2 \subset \mathbb{R}^d \end{array}$$

Proof. Robinson (See 1999, sections 5.6 and 5.7) or Coudène (2016, section 8.4) among others □

Thus, if a fixed point x^* is hyperbolic, we can analyze its stability properties by investigating the induced linear homogeneous difference equation and, in particular, the eigenvalues of the matrix of partial derivatives.

Chapter 2

Univariate Difference Equations with Constant Coefficients

While the introduction and the notion of stability was somewhat abstract, we focus in this chapter on concrete difference equations and examples. In particular, we study nonautonomous or homogenous linear difference equations of dimension one ($d = 1$) and order $p \geq 1$ with constant coefficients:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + z_t, \qquad \phi_p \neq 0, \tag{2.1}$$

where $\phi_1, \ldots, \phi_p$ are given constant real numbers. The variable $z_t \in \mathbb{R}$ represents the nonautonomous part of the equation which influences the evolution of x_t over time. Its values are given from outside the system. Thus, z_t is called exogenous, independent or forcing variable. It will be shown that for this type of difference equations explicit solution formulas are available.

2.1 First Order Difference Equation

As a starting point and motivation of the analysis consider the simplest case, namely the first order ($p = 1$) affine nonhomogeneous equation:

$$x_t = \phi x_{t-1} + z_t, \qquad \phi \neq 0. \tag{2.2}$$

To this nonhomogeneous equation corresponds a first order linear homogenous equation:

$$x_t = \phi x_{t-1}. \tag{2.3}$$

Starting in period 0 at some arbitrary initial value $x_0 = x$, all subsequent values can be recursively computed by iteratively inserting into the difference

equation (2.3)

$$\begin{aligned} x_0 &= x \\ x_1 &= \phi x_0 = \phi x \\ x_2 &= \phi x_1 = \phi^2 x \\ &\dots \\ x_t &= \phi x_{t-1} = \phi^t x. \end{aligned}$$

This suggests to take

$$x_t = \varphi(t, x) = \phi^t x \tag{2.4}$$

as the *general solution* of the first order linear homogenous difference equation (2.3). Actually, equation (2.4) provides a whole family of solutions indexed by the starting value $x \in \mathbb{R}$. To each value of x, there corresponds a trajectory $x_t = \varphi(t, x) = \phi^t x$ with corresponding forward orbit $\mathcal{O}^+(x) = \{x, \phi x, \phi^2 x, \dots\}$.

Note that the trajectories of two different solutions $\varphi(t, c_1)$ and $\varphi(t, c_2)$, $c_1 \neq c_2$, cannot cross. For suppose there exists τ such that $\varphi(\tau, c_1) = \varphi(\tau, c_2)$, then $\phi^\tau c_1 = \phi^\tau c_2$. This implies $c_1 = c_2$ because $\phi \neq 0$. This contradicts the assumption $c_1 \neq c_2$.

The parameter x can be pinned down by using a single *boundary condition*. A simple form of such a boundary condition requires, for example, that x_t takes a particular value c in some period t_0. Thus, we require that $\varphi(t_0, x) = c$. In this case we speak of an *initial value problem*. The value of x can then be retrieved by solving the equation $x_{t_0} = \phi^{t_0} x = c$ for x. This leads to $x = \phi^{-t_0} c$. The solution may then be written as

$$x_t = \varphi(t, x) = \phi^{t-t_0} c = \varphi(t - t_0, c).$$

Note that the solution depends on $t - t_0$ and not on t or t_0 separately. In many instances we are given the value at $t_0 = 0$ so that $x = c$.

Because $\phi \neq 0$, we can iterate equation (2.3) also backwards in time: $x_{-t} = \phi^{-1} x_{-t+1}$, $t = 1, 2, \dots$ Thus, the solution $x_t = \varphi(t, x) = \phi^t x$ holds for all integers $t \in \mathbb{Z}$ with orbit $\mathcal{O}(x) = \{\dots, \phi^{-2} x, \phi^{-1} x, x, \phi x, \phi^2 x, \dots\}$.

Suppose that we are given two solutions of the homogenous equation, $(x_t^{(1)})$ and $(x_t^{(2)})$. Then it is easy to verify that any linear combination of the two solutions, $a_1(x_t^{(1)}) + a_2(x_t^{(2)})$, $a_1, a_2 \in \mathbb{R}$, is also a solution. This implies that the set of all solutions or orbits of the homogenous equation forms a *linear space* or *vector space*. In order to find out the dimension of this linear space and its algebraic structure, it is necessary to introduce the following three important definitions.

Definition 2.1 (Linear Dependence, Linear Independence)**.** The r sequences $(x^{(1)}), (x^{(2)}), \ldots, (x^{(r)})$ with $r \geq 2$ are said to be *linearly dependent* for $t \geq t_0$ if there exist constants $a_1, a_2, \ldots, a_r \in \mathbb{R}$, not all zero, such that

$$a_1 x_t^{(1)} + a_2 x_t^{(2)} + \cdots + a_r x_t^{(r)} = 0, \qquad \text{for all } t \geq t_0.$$

This definition is equivalent to saying that there exists a nontrivial linear combination of the solutions which is zero. If the solutions are not linearly dependent, they are said to be *linearly independent*.

Definition 2.2 (Fundamental Set of Solutions)**.** A set of r linearly independent solutions of the homogenous equation is called a *fundamental set* of solutions.

Definition 2.3 (Casarotian Matrix)**.** The *Casarotian matrix* $\mathcal{C}(t)$ of $(x^{(1)})$, $(x^{(2)}), \ldots, (x^{(r)})$ with $r \geq 1$ is defined as

$$\mathcal{C}(t) = \begin{pmatrix} x_t^{(1)} & x_t^{(2)} & \cdots & x_t^{(r)} \\ x_{t+1}^{(1)} & x_{t+1}^{(2)} & \cdots & x_{t+1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{t+r-1}^{(1)} & x_{t+r-1}^{(2)} & \cdots & x_{t+r-1}^{(r)} \end{pmatrix}.$$

These definitions allow us to the tackle the issue of the dimension of the linear space given by all solutions to the homogenous first order linear difference equation.

Theorem 2.1 (Dimension of Linear First Order Equation)**.** The set of solutions to the homogenous first order linear difference equation (2.3) is a linear space of dimension one.

Proof. Suppose we are given two linearly independent solution $(x^{(1)})$ and $(x^{(2)})$. Then according to Definition 2.1, for all constants a_1 and a_2, not both equal to zero,

$$\begin{aligned} a_1 x_t^{(1)} + a_2 x_t^{(2)} &\neq 0 \\ a_1 x_{t+1}^{(1)} + a_2 x_{t+1}^{(2)} &\neq 0. \end{aligned}$$

Inserting in the second inequality $\phi x_t^{(1)}$ for $x_{t+1}^{(1)}$ and $\phi x_t^{(2)}$ for $x_{t+1}^{(2)}$ leads to

$$\begin{aligned} a_1 x_t^{(1)} + a_2 x_t^{(2)} &\neq 0 \\ a_1 \phi x_t^{(1)} + a_2 \phi x_t^{(2)} &\neq 0 \end{aligned}$$

or equivalently

$$\begin{pmatrix} x_t^{(1)} & x_t^{(2)} \\ \phi x_t^{(1)} & \phi x_t^{(2)} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \neq 0$$

Since this must hold for any a_1, a_2, not both equal to zero, the determinant of the Casarotian matrix (see Definition 2.3)

$$\det \mathcal{C}(t) = \det \begin{pmatrix} x_t^{(1)} & x_t^{(2)} \\ \phi x_t^{(1)} & \phi x_t^{(2)} \end{pmatrix}$$

must be nonzero. However, $\det \mathcal{C}(t) = \phi x_t^{(1)} x_t^{(2)} - \phi x_t^{(1)} x_t^{(2)} = 0$. This is a contradiction to the initial assumption. Thus, there can only be one independent solution. □

The only independent solution is therefore given by (2.4). In Section 2.4 we will give a general proof and show that the dimension of the linear space generated by the solutions to the homogenous equation of order p is p.

Consider now two solutions of the nonhomogeneous difference equation (2.2), $(x_t^{(1)})$ and $(x_t^{(2)})$, then, as can be easily verified, $(x_t^{(1)}) - (x_t^{(2)})$ satisfies the homogenous equation (2.3). This fact is called the *superposition principle*.[1] The superposition principle implies that $x_t^{(1)} - x_t^{(2)} = \phi^t x$ which leads to the following theorem.

Theorem 2.2 (Superposition Principle)**.** Every solution, $\varphi(t, x)$, of the first order nonhomogeneous affine difference equation (2.2) can be represented as the sum of the general solution of homogenous equation (2.3), $x_t^{(g)}$, and a particular solution to the nonhomogeneous equation, $x_t^{(p)}$:

$$x_t = x_t^{(g)} + x_t^{(p)}. \tag{2.5}$$

The proof of this theorem is easily established and is left as an exercise to the reader. In the case of a first order equation $x_t^{(g)} = \phi^t x$. The Superposition Principle then implies that the solution of the first order equation is given by:

$$x_t = \phi^t x + x_t^{(p)}.$$

[1]The superposition principle means that the net response of x_t caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually. In the first order case one stimulus comes from the general solution to the homogeneous equation, the other from the particular solution to the nonhomogeneous equation.

Below we will discuss how to obtain a particular solution. We will also see subsequently that this principle extends to higher order equations and linear systems. The Superposition Principle thus delivers a general recipe for solving linear difference equations in three steps:

1. Find the general solution of the homogeneous equation $x^{(g)}$. This is usually a technical issue that can be resolved mechanically.

2. Find a particular solution to the nonhomogeneous equation $x^{(p)}$. This step is usually more involved and requires additional (economic) arguments. For example, we might argue that if the forcing variable z_t remains bounded then also x_t should remain bounded.

3. The Superposition Principle (see Theorem 2.2) then delivers the general solution of the nonhomogeneous equation as the sum of $x^{(g)}$ and $x^{(p)}$. However, this solution still depends through $x^{(g)}$ on some constants. To pin down the solution uniquely and therefore solving the boundary value problem requires additional conditions. These conditions can come in the form of initial values (starting values) or in the form of requirements that the solution must obey some qualitative feature. A typical feature in this context is boundedness, a condition which usually can be given an economic underpinning.

Before continuing with the theoretical analysis consider the following basic example.

Amortization of a Loan

One of the simplest setting in economics where a difference equation arises naturally, is compound interest calculation. Take, for example, the evolution of debt. Denote by D_t the debt outstanding at the beginning of period t, then the debt in the subsequent period $t+1$, D_{t+1}, is obtained by the simple accounting rule:

$$D_{t+1} = D_t + rD_t - Z_t = (1+r)D_t - Z_t \tag{2.6}$$

where rD_t is the interest accruing at the end of period t. Here we are using for simplicity a constant interest rate r. The debt contract is serviced by paying some amount Z_t at the end of period t. This payment typically includes a payment for the interest and a repayment of the principal. Equation (2.6) constitutes a linear nonhomogeneous first order difference equation with $\phi = 1 + r$.

Given the initial debt at the beginning of period 0, D_0, the amount of debt outstanding in subsequent periods can be computed recursively using the accounting rule (2.6):

$$\begin{aligned} D_1 &= (1+r)D_0 - Z_0 \\ D_2 &= (1+r)D_1 - Z_1 = (1+r)^2 D_0 - (1+r)Z_0 - Z_1 \\ &\dots \\ D_{t+1} &= (1+r)^{t+1} D_0 - Z_t - (1+r)Z_{t-1} - \dots - (1+r)^t Z_0 \\ &= (1+r)^{t+1} D_0 - \sum_{i=0}^{t} (1+r)^i Z_{t-i} \end{aligned}$$

Note how D_{t+1} is determined as the sum of two parts: $(1+r)^{t+1}D_0$ and $-\sum_{i=0}^{t}(1+r)^i Z_{t-i}$. The first expression thereby corresponds to the general solution of the homogeneous equation and the second one to a particular solution of the nonhomogeneous equation in accordance with Theorem 2.2.[2] As the initial value of the debt is given, this value naturally pins down the parameter c to equal D_0.

If the repayments Z_t are constant over time and equal to Z, as is often the case, we can bring Z outside the summation sign and use the formula for geometric sums to obtain:

$$D_{t+1} = (1+r)^{t+1} D_0 - \left((1+r)^{t+1} - 1\right) \frac{Z}{r}.$$

Suppose that the debt must be completely repaid by the beginning of period $T+1$, then the corresponding constant period payment Z can be calculated by setting $D_{T+1} = 0$ in the above equation and solving for Z.[3] This gives:

$$Z = \frac{r}{1 - (1+r)^{-T-1}} D_0.$$

Note that the payment Z required to pay back the debt diminishes with T. If T approaches infinity Z equals rD_0. In this case the payment is just equal to interest accruing in each period so that there is no repayment of the principal. In this case the debt is never paid back and equals the initial debt D_0 in each period. If the payment Z exceeds rD_0, the debt is repaid in a finite amount of time.

[2] The reader is invited to check that the second expression is really a solution to the nonhomogeneous equation.

[3] The repayments are, of course, only constant as long as $D_t > 0$. Once the debt is paid back fully, payments cease and $Z = 0$ from then on.

Suppose that instead of requiring that the debt must be zero at some point in time (including infinity), we impose the condition that the present discounted value of the debt must be non-positive as T goes to infinity:

$$\lim_{T\to\infty} \frac{D_{T+1}}{(1+r)^{T+1}} \leq 0. \tag{2.7}$$

This condition is referred to as the *no Ponzi game* (NPG) condition in economics. A Ponzi game is a scheme where all principal repayments and interest payments are rolled over perpetually by issuing new debt.[4] If the above limit is positive, the borrower would be able to extract resources (in present value terms) from the lenders (See O'Connell and Zeldes (1988) and the literature cited therein for an assessment of the significance of the NPG condition in economics). Given the difference equation for the evolution of debt, the NPG condition with constant payment per period is equivalent to:

$$\lim_{T\to\infty} \frac{D_{T+1}}{(1+r)^{T+1}} = \lim_{T\to\infty} D_0 - \left(1-(1+r)^{-T-1}\right)\frac{Z}{r} = D_0 - \frac{Z}{r} \leq 0$$

which implies that $Z \geq rD_0$. Thus, the NPG condition holds if the constant repayments Z are at least as great as the interest.

2.2 Solutions of First Order Affine Difference Equations

This section discusses a more systematic way of finding a particular solution to the first order affine difference equation (2.2). For this purpose insert recursively equation (2.2) into itself:

$$\begin{aligned}
x_t &= \phi x_{t-1} + z_t \\
x_t &= \phi(\phi x_{t-2} + z_{t-1}) + z_t = \phi^2 x_{t-2} + \phi z_{t-1} + z_t \\
&\cdots \\
x_t &= \phi^t x_0 + \phi^{t-1} z_1 + \phi^{t-2} z_2 + \cdots + \phi z_{t-1} + z_t \\
&= \phi^t x_0 + \sum_{j=0}^{t-1} \phi^j z_{t-j}
\end{aligned}$$

[4]Charles Ponzi was an Italian immigrant who promised to pay exorbitant returns to investors out of an ever-increasing pool of deposits. A historic account of Ponzi games can be found in Kindleberger (1978).

Taking the absolute value of the difference between x_t and the second term of the right hand side of the equation leads to:

$$\left| x_t - \sum_{j=0}^{t-1} \phi^j z_{t-j} \right| = \left| \phi^t x_0 \right| = |\phi^t| \, |x_0|$$

When there is a starting period as in the example of the amortization of a loan, say period 0 without loss of generality, we stop the backwards iteration at this period and take $x_t^{(p)} = \sum_{j=0}^{t-1} \phi^j z_{t-j}$ as the particular solution. However in many instances there is no natural starting period so that it makes sense to continue the above iteration into the infinite remote past. Given that $|\phi^t|$ vanishes as $t \to \infty$ if and only if $|\phi| < 1$, this suggests to consider

$$x_t^{(b)} = \sum_{j=0}^{\infty} \phi^j z_{t-j} \tag{2.8}$$

as a particular solution to the equation (2.2). The superscript (b) indicates that the solution was obtained by iterating the difference equation backward in time. For this to be a meaningful choice, the infinite sum must be well–defined. This is, for example, the case if $\{z_t\}$ is a bounded sequence. In particular, if z_t is constant and equal to z, the above particular solution becomes

$$x_t^{(b)} = \sum_{j=0}^{\infty} \phi^j z = \frac{z}{1-\phi}, \qquad |\phi| < 1,$$

which is just the steady state solution described in Section 1.2.

The requirement that z_t remains bounded can, for example, be relaxed if z_t itself satisfies the homogenous difference equation $z_t = \psi z_{t-1}$ which implies that $z_t = \psi^t c$ for some $c \neq 0$. For $|\psi| > 1$, z_t is unbounded. Inserting this into equation (2.8) then leads to

$$x_t^{(b)} = \sum_{j=0}^{\infty} \phi^j \psi^{t-j} c = \psi^t \sum_{j=0}^{\infty} \left(\frac{\phi}{\psi} \right)^j c.$$

The infinite sum converges if and only if $|\phi/\psi| < 1$. This shows that besides the stability condition $|\phi| < 1$, some additional requirements with respect to the sequence of the exogenous variable are necessary to render $x_t^{(b)}$ in equation (2.8) a meaningful particular solution. Usually, we assume that z_t is bounded.

Consider next the case $|\phi| > 1$. In this situation the above iteration is no longer successful because $x_t^{(b)}$ in equation (2.8) is not well–defined even when

z_t is constant. A way out of this problem is to consider the iteration forward in time instead of backward in time:

$$\begin{aligned} x_t &= \phi^{-1}x_{t+1} - \phi^{-1}z_{t+1} \\ &= \phi^{-1}\left(\phi^{-1}x_{t+2} - \phi^{-1}z_{t+2}\right) - \phi^{-1}z_{t+1} = \phi^{-2}x_{t+2} - \phi^{-2}z_{t+2} - \phi^{-1}z_{t+1} \\ &\dots \\ &= \phi^{-h}x_{t+h} - \phi^{-1}\sum_{j=1}^{h}\phi^{-j+1}z_{t+j} \qquad \text{for } h \geq 1. \end{aligned}$$

Taking the absolute value of the difference between x_t and the second term on the right hand side of the equation leads to:

$$\left|x_t + \phi^{-1}\sum_{j=1}^{h}\phi^{-j+1}z_{t+j}\right| = \left|\phi^{-h}x_{t+h}\right| = |\phi^{-h}|\,|x_{t+h}|.$$

As the economy is expected to live forever, there is no end period and the forward iteration can be carried out indefinitely into the future. Because $|\phi| > 1$, the right hand side of the equation converges to zero as $h \to \infty$, provided that x_{t+h} remains bounded. This suggests the following particular solution:

$$x_t^{(f)} = -\phi^{-1}\sum_{j=1}^{\infty}\phi^{-j+1}z_{t+j}, \qquad |\phi| > 1, \tag{2.9}$$

where the superscript (f) indicates that the solution was obtained by iterating the difference equation forward in time. For this to be a meaningful choice, the infinite sum must be well-defined. This will be guaranteed if, for example, z_t remains bounded.

When $|\phi| = 1$ neither the backward nor the forward iteration strategy leads to a sensible particular solution even when z_t is constant and equal to $z \neq 0$. Either an equilibrium point does not exist as in the case $\phi = 1$ or the equilibrium point exists as is the case for $\phi = -1$, but x_t oscillates forever between x_0 and $-x_0 + z$ so that the equilibrium point is unstable. Most of the time, we restrict ourself to the hyperbolic case and disregard the case $|\phi| = 1$.

To summarize, assuming that $\{z_t\}$ is bounded, the first order linear difference equation (2.2) led us to consider the following two representations of

the general solutions:

$$x_t = \phi^t x + x_t^{(b)}, \qquad \text{whereby} \quad x_t^{(b)} = \sum_{j=0}^{\infty} \phi^j z_{t-j}$$

$$x_t = \phi^t x + x_t^{(f)}, \qquad \text{whereby} \quad x_t^{(f)} = -\phi^{-1} \sum_{j=1}^{\infty} \phi^{-j+1} z_{t+j}$$

Note that these equations imply that $x = x_0 - x_0^{(b)}$, respectively $x = x_0 - x_0^{(f)}$. Depending on the value of ϕ, we can distinguish the following three cases:

$|\phi| < 1$: the backward solution is asymptotically stable in the sense that x_t approaches $x_t^{(b)}$ as $t \to \infty$. Any deviation of x_t from $x_t^{(b)}$ vanishes over time, irrespective of the value chosen for x, because $\lim_{t\to\infty} |x_t - x_t^{(b)}| = \lim_{t\to\infty} |\phi^t x| = |x| \lim_{t\to\infty} |\phi^t| = 0$. The forward solution, usually, makes no sense because $x_t^{(f)}$ is not well–defined despite the fact that the forcing variable z_t is bounded.

$|\phi| > 1$: both solutions have an explosive behavior due to the term ϕ^t. Even small deviations from either $x_t^{(b)}$ or $x_t^{(f)}$ will grow without bounds. There is, however, one and only one solution which remains bounded. It is given by $x = 0$ which implies that x_t always equals its equilibrium value $x_t^{(f)}$.

$|\phi| = 1$: neither the backward nor the forward solution converge for constant $z_t \neq 0$.

Which solution is appropriate depends on the nature of the economic problem at hand. In particular, the choice of the boundary condition requires some additional thoughts and cannot be determined on general grounds. As the exercises below demonstrate, the nature of the expectations formation mechanism is crucial in this respect.

2.3 Examples of First Order Linear Difference Equations

2.3.1 The simple Cobweb Model

The Cobweb model, originally introduced by Moore (1914) to analyze the cyclical behavior of agricultural markets, was one of the first dynamic models

in economics. It inspired an enormous empirical as well as theoretical literature. Its analysis culminated in the introduction of rational expectations by Muth (1961). The model, in its simplest form, analyzes the short–run price fluctuations in a single market where, in each period, the price level is determined to equate demand and supply. Their logarithms are denoted by D_t and S_t, respectively. The good exchanged on this market is not storable and is produced with a fixed production lag of one period. The supply decision of producers in period $t-1$ is based on the price they expect to get for their product when the enter the market in period t. Denoting the logarithm of the price level in period t by p_t and assuming a negatively sloped demand curve and a positively sloped supply curve, the simple Cobweb model can be summarized by the following four equations:[5]

$$\begin{aligned} D_t &= -\beta p_t, & \beta > 0 \qquad & \text{(demand)} \\ S_t &= \gamma p_t^e + u_t, & \gamma > 0 \qquad & \text{(supply)} \\ S_t &= D_t & & \text{(market clearing)} \\ p_t^e &= p_{t-1} & & \text{(expectations formation)} \end{aligned}$$

where u_t denotes a supply shock. In agricultural markets u_t typically represents weather conditions or breeding conditions in the case of animals.

Given the naive expectations formation, $p_t^e = p_{t-1}$, the model can be solved to yield a linear first order difference equation in p_t:

$$p_t = -\frac{\gamma}{\beta} p_{t-1} - \frac{u_t}{\beta} = \phi p_{t-1} + z_t \tag{2.10}$$

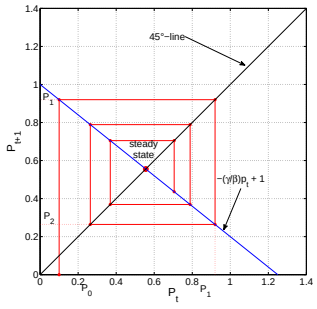
where $\phi = -\frac{\gamma}{\beta}$ and $z_t = -\frac{u_t}{\beta}$. Due to the negative value of ϕ, the price oscillates: high prices tend to be followed by low prices which are again followed by high prices. These price oscillations translate into corresponding quantity oscillations. If u_t is independent of time and equal to u, the equilibrium price p^* of the Cobweb model can be computed as follows:

$$p^* = -\frac{\gamma}{\beta} p^* - \frac{u}{\beta} \qquad \Rightarrow \qquad p^* = \frac{-u}{\beta + \gamma} \tag{2.11}$$

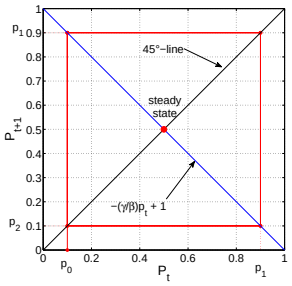
If demand is relatively elastic compared to supply, i.e. if $\beta > \gamma$, $0 < \phi < 1$ and the equilibrium p^* is asymptotically stable.

The Figure 2.1 depicts several possible cases depending on the relative slopes of supply and demand. In the first panel $\phi = -0.8$ so that we have an asymptotically stable equilibrium. Starting at p_0, the price approaches the steady state by oscillating around it. In the second panel $\beta = \gamma$ implying $\phi =$

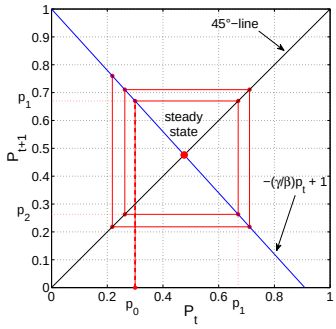
[5]The logarithm of the price level is taken to ensure a positive price level.



(a) convergence ($\beta = 1$ and $\gamma = 0.8$)



(b) oscillating ($\beta = 1$ and $\gamma = 1$)



(c) exploding ($\beta = 1$ and $\gamma = 1.1$)

Figure 2.1: Price dynamics in the cobweb model

-1 so that, independently of the starting value, the price oscillates forever between p_0 and p_1. In the third panel, we have an unstable equilibrium. Starting at $p_0 \neq p^*$, p_t diverges.

2.3.2 The Solow Growth Model

Although this chapter only deals with linear or affine difference equations, the local behavior of nonlinear difference equations can be studied by linearizing the difference equation around its steady state and applying Theorem 1.2. A classic example in this respect is the famous Solow growth model (see Solow (1956)). A simple version of this model describes a closed economy with no technical progress. Output in period t, denoted by Y_t, is produced with two essential production factors: capital, K_t, and labor, L_t. Production possibilities of this economy in period t are described by a *neoclassical production function* $Y_t = F(K_t, L_t)$. This production function is defined on the nonnegative orthant of $\mathbb{R}^2$ and is characterized by the following properties:

- F is twice continuously differentiable;

- strictly positive marginal products, i.e. $\frac{\partial F(K,L)}{\partial K} > 0$ and $\frac{\partial F(K,L)}{\partial L} > 0$;
- diminishing marginal products, i.e. $\frac{\partial^2 F(K,L)}{\partial K^2} < 0$ and $\frac{\partial^2 F(K,L)}{\partial L^2} < 0$;
- F has constant returns-to-scale, i.e. $F(\lambda K, \lambda L) = \lambda F(K, L)$ for all $\lambda > 0$;
- F satisfies the *Inada* conditions:

$$\begin{aligned} \lim_{K\to\infty} \frac{\partial F(K,L)}{\partial K} &= 0, \qquad \lim_{L\to\infty} \frac{\partial F(K,L)}{\partial L} = 0, \\ \lim_{K\to 0} \frac{\partial F(K,L)}{\partial K} &= \infty, \qquad \lim_{L\to 0} \frac{\partial F(K,L)}{\partial L} = \infty. \end{aligned}$$

The Inada conditions are usually not listed among the properties of a neoclassical production function, however, they turn out to be necessary to guarantee a strictly positive steady state. The classic example for a production function with these properties is the Cobb-Douglas production function: $F(K, L) = AK^{(1-\alpha)}L^{\alpha}$, $A > 0$, $0 < \alpha < 1$. The above properties have two important implications summarized by the following lemmata.

Lemma 2.1 (Essential Inputs)**.** Let F be a neoclassical production function as described above then both inputs are essential, i.e. $F(K, 0) = 0$ and $F(0, L) = 0$.

Proof. Suppose that $Y \to \infty$ as $K \to \infty$ then L'Hôpital's rule together with the Inada conditions imply

$$\lim_{K\to\infty} \frac{Y}{K} = \lim_{K\to\infty} \frac{\partial Y/\partial K}{1} = \lim_{K\to\infty} \frac{\partial Y}{\partial K} = 0.$$

If on the other hand, Y remains finite when $K \to \infty$, we immediately also get

$$\lim_{K\to\infty} \frac{Y}{K} = 0.$$

The constant returns to scale assumption then implies that, for $L > 0$ fixed,

$$\lim_{K\to\infty} \frac{Y}{K} = \lim_{K\to\infty} F(1, L/K) = F(1, 0) = 0.$$

Using the constant returns to scale assumption again, we derive

$$F(K, 0) = KF(1, 0) = 0.$$

Thus, capital is essential. The proof that L is essential is analogous. □

Lemma 2.2. Let F be a neoclassical production function as described above then output goes to infinity if either input goes to infinity, holding the other input fixed.

Proof. Omitting the time subscripts, define k as the capital intensity, i.e. $k = K/L$. The assumption of constant returns to scale then implies that

$$F(K, L) = LF(K/L, 1) = Lf(k) = Kf(k)/k$$

where $f(k) = F(k, 1)$. Holding $K > 0$ fixed, L'Hôpital's rule implies

$$\lim_{L\to\infty} F(K, L) = K \lim_{k\to 0} \frac{f(k)}{k} = K \lim_{k\to 0} f'(k) = \infty$$

where we have used the result that capital is essential (see the previous Lemma), i.e. that $f(0) = 0$. The last equality is a consequence of the Inada conditions. The proof for $\lim_{K\to\infty} F(K, L) = \infty$, $L > 0$ fixed, is analogous. □

Output can be used either for consumption, C_t, or investment, I_t:

$$Y_t = C_t + I_t. \tag{2.12}$$

The economy saves a constant fraction $s \in (0, 1)$ of the output. Because saving equals investment in a closed economy, we have

$$I_t = sY_t. \tag{2.13}$$

Investment adds to the existing capital stock which depreciates in each period at a constant rate $\delta \in (0, 1)$:

$$K_{t+1} = (1 - \delta)K_t + I_t = (1 - \delta)K_t + sY_t = (1 - \delta)K_t + sF(K_t, L_t) \tag{2.14}$$

Whereas capital is a reproducible factor of production, labor is a fixed factor of production which is assumed to grow at the exogenously given constant rate $\mu > 0$:

$$L_{t+1} = (1 + \mu)L_t, \qquad L_0 > 0 \text{ given.} \tag{2.15}$$

Starting in period 0 with some positive capital $K_0 > 0$, the system consisting of the two difference equations (2.14) and (2.15) completely describes the evolution of the economy over time. A first inspection of the two equations immediately reveals that both labor and capital tend to infinity. Indeed, as $\mu > 0$ labor grows without bound implying according to Lemma 2.2 that output also grows without bound. This is not very revealing if one is looking for steady states and is interested in a stability analysis. In such a situation

it is often advisable to look at the ratio of the two variables, in our case at K/L. This has several advantages. First, the dimension of the system is reduced to one. Second, and more importantly, the singularity at infinity is, at least in the linear case, eliminated.[6] Third, these ratios often have a clear economic meaning making the economic interpretation of the results more comprehensible.

We apply this device to the Solow model as described by equations (2.14) and (2.15). Thus, dividing equation (2.14) by L_{t+1} and making use of the constant returns to scale assumption results in the fundamental equation of the Solow model:

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{1-\delta}{1+\mu}k_t + \frac{s}{1+\mu}f(k_t) = g(k_t) \tag{2.16}$$

where $k_t = \frac{K_t}{L_t}$ is known as the *capital intensity* and $f(k_t) = F\left(\frac{K_t}{L_t}, 1\right)$. The production function in terms of capital intensity inherits the properties of the original production function: $f'(k) > 0$ and $f''(k) < 0$, and $\lim_{k\to\infty} f'(k) = 0$ and $\lim_{k\to 0} f'(k) = \infty$. These properties also carry over to g with the exception that $\lim_{k\to\infty} g'(k) = (1-\delta)/(1+\mu) < 1$.

The economy starts in period zero with an initial capital intensity $k_0 > 0$. The nonlinear first order difference equation (2.16) together with the initial condition uniquely determines the evolution of the capital intensity over time, and consequently of all other variables in the model. Hence, by solving the difference equation (2.16) for k_t all other variables of the model are determined as well.

Proposition 2.1. Given the assumptions of the Solow model, the fundamental Solow equation (2.16) has exactly two steady states $k^* = 0$ and $k^* > 0$.

Proof. The steady states must satisfy the nonlinear equation:

$$k^* = g(k^*).$$

This equation implies

$$k^* = \frac{s}{\mu+\delta}f(k^*). \tag{2.17}$$

[6] Technically speaking, this induces a new difference equation on the projective space. In the two dimensional case, the projective space is defined as the set of rays through the origin. As each ray crosses the unit circle twice, an equivalent definition is given as the unit circle where opposite points are not distinguished. See Colonius and Kliemann (2014, chapter 4) for details.

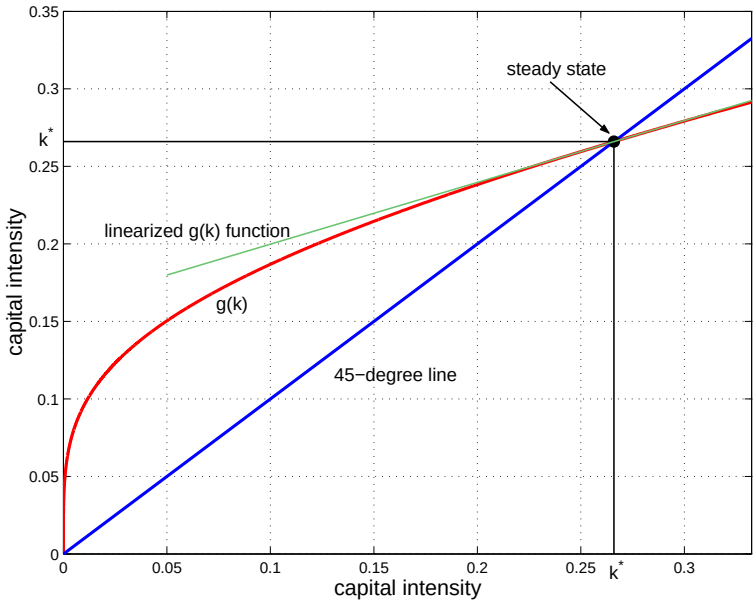


Figure 2.2: Capital Intensity in the Solow Model

As $f(0) = 0$, $k^* = 0$ is a steady state. From Figure 2.2 it becomes clear that the Inada conditions together with concavity ensure the existence of a unique strictly positive steady state $k^* > 0$. In particular, the properties $g(0) = 0$, $\lim_{k\to 0} g'(k) = \infty$, $\lim_{k\to\infty} g'(k) = \frac{1-\delta}{1+\mu} < 1$, and $g'(k) > 0$ ensure that the function g is sufficiently steep at the origin and becomes eventually flat enough to cross the 45-degree line once from above. □

The steady state $k^* = 0$ is of no economic significance. The asymptotic stability of $k^* > 0$ is easily established by observing that k_t is monotonically increasing for $k_0 \in (0, k^*)$ and monotonically decreasing for $k_0 > k^*$. Thus, k_t converges monotonically to k^* independently of the initial value $k_0 > 0$. This shows that $k^* > 0$ is attracting. Monotonicity also implies stability because for all $\varepsilon > 0$, taking $\delta_\varepsilon = \varepsilon$, $|k_0 - k^*| < \delta_\varepsilon$ implies $|k_t - k^*| < \varepsilon$ for all $t \geq 0$. Thus, $k^* > 0$ is stable and therefore asymptotically stable. Another way to establish this conclusion is to invoke Theorem 1.1. This also holds globally when $\mathbf{X}$ is restricted to $\mathbf{X} = \mathbb{R}^+$.

This fact can also be established by invoking Theorem 1.2. To do so, we linearize equation (2.16) around the steady state $k^* > 0$. This amounts to take a first order Taylor approximation:

$$k_{t+1} \approx k^* + \left.\frac{\partial g(k)}{\partial k}\right|_{k=k^*} (k_t - k^*) \tag{2.18}$$

We can therefore study the local behavior of the nonlinear difference equation (2.16) around the steady state $k^* > 0$ by investigating the properties of the first order homogenous difference equation:

$$k_{t+1} - k^* = \phi\,(k_t - k^*) \tag{2.19}$$

where $0 < \phi = \left.\frac{\partial g(k)}{\partial k}\right|_{k=k^*}$.

Proposition 2.2. $0 < \phi = \left.\frac{\partial g(k)}{\partial k}\right|_{k=k^*} < 1$.

Proof. Note that $g'(k) > 0$ for all $k > 0$. Concavity of g implies that $g(k) - k^* \leq g'(k^*)(k - k^*)$ for all $k > 0$. Take $k < k^*$, then $g(k) > k$. Thus, $g(k) - k^* < g'(k^*)(g(k) - k^*) < 0$. Noting that $g(k) - k^* < 0$ implies that $g'(k^*) < 1$. □

Starting in period zero with an initial capital intensity $k_0 > 0$, the approximate solution to this initial value problem near the steady state is:

$$k_t = k^* + \phi^t(k_0 - k^*)$$

As $0 < \phi < 1$, the steady state k^* is asymptotically stable, even exponentially so.

2.3.3 A Model of Equity Prices

Consider an economy where investors have just two assets at their disposal. The first one is a riskless government bond which pays a constant interest rate $r > 0$ in each period. The second one is a common share which gives the owner the right to a known dividend stream per share. The problem is to determine the share price p_t as a function of the future dividend stream $(d_{t+h})_{h=0,1,\ldots}$ and the interest rate r. As we abstract from uncertainty in this example, arbitrage ensures that the return on both investments must be equal. Given that the return on the investment in the share consists of the dividend payment d_t plus the expected price change $p^e_{t+1} - p_t$, this arbitrage condition yields:

$$r = \frac{d_t + p^e_{t+1} - p_t}{p_t} \qquad \Leftrightarrow \qquad p^e_{t+1} = (1+r)p_t - d_t \tag{2.20}$$

where p^e_{t+1} denotes the price expected to prevail in the next period. Assuming that expectations of the investors are *rational* which is equivalent to assuming perfect foresight in the context of no uncertainty, the above arbitrage equation turns into a simple first order difference equation:

$$p_{t+1} = (1+r)p_t - d_t \tag{2.21}$$

with $\phi = 1 + r$ and $z_t = -d_{t-1}$. Note that we are given no initial condition. Instead, the purpose is to find a starting price (initial value) p_0.

Whereas the general solution to the homogeneous equation is easily found to be $x_t^{(g)} = \phi^t c_f$, for some $c_f \in \mathbb{R}$, the search for an appropriate particular solution to the nonhomogeneous equation requires additional considerations. Because $\phi = 1 + r > 1$, we can disregard the backward solution as the infinite sum will not converge for a constant dividend stream. Thus, we turn to the forward solution $x_t^{(f)} = -\phi^{-1} \sum_{j=1}^{\infty} \phi^{-j+1} z_{t+j}$ (see equation (2.9)). We therefore envision the following general solution to the difference equation (2.21):

$$p_t = (1+r)^t c_f + x_t^{(f)} = \left(p_0 - x_0^{(f)}\right)(1+r)^t + x_t^{(f)} \tag{2.22}$$

where $x_t^{(f)} = (1+r)^{-1} \sum_{j=0}^{\infty} (1+r)^{-j} d_{t+j}$. Note that the forward solution is only well-defined if the infinite sum converges. A sufficient condition for this to happen is the existence of a finite index j_0 such that $|d_{t+j}/(1+r)^j| < M^j$, for $j > j_0$ and some $M < 1$. This is guaranteed, in particular, by a constant dividend stream $d_{t+j} = d$, for all $j = 0, 1, 2, \ldots$

The term $(1+r)^t c_f$ is usually called the bubble term because its behavior is unrelated to the dividend stream; whereas the term $x_t^{(f)}$ is referred to as the fundamentals because it is supposed to reflect the "intrinsic value" of the share.

Remember that the purpose is to figure out the initial price of a share p_0. Take period 0 to be the current period and suppose that $c_f = p_0 - x_0^{(f)} > 0$. This means that the current stock price is higher than what can be justified by the future dividend stream. According to the arbitrage equation (2.20) this high price (compared to the dividend stream) can only be justified by an appropriate capital gain, i.e. an appropriate expected price increase in the next period. This makes the price in the next period even more different from the fundamentals which must be justified by an even greater capital gain in the following period, and so on. In the end, the bubble term takes over and the share price becomes almost unrelated to the dividend stream. This situation is, however, not sustainable in the long run.[7] Therefore, the only reasonable current share price p_0 is $x_0^{(f)}$ which implies that $c_f = 0$. This effectively eliminates the bubble term and is actually the only nonexplosive solution. Thus, we have a unique (determinate) rational equilibrium solution. This solution is

$$p_t = x_t^{(f)} = (1+r)^{-1} \sum_{j=0}^{\infty} (1+r)^{-j} d_{t+j} \tag{2.23}$$

[7] A similar argument applies to the case $c_f = p_0 - x_0^{(f)} < 0$.

Thus, the price of a share always equals the present discounted value of the corresponding dividend stream. Such a solution is reasonable in a situation with no uncertainty and no information problems.

If dividends are expected to be constant and equal to d, the above equation simplifies to $p_t = d/r$ or $p_t/d = 1/r$. p_t/d is known as the price dividend ratio and plays an important role in financial markets. In particular, it is interpreted as an indicator for the over- respectively undervaluation of a share.

The above solution implies that the price immediately responds to any change in the expected dividend stream. The effect of a change in $d_{t+h}, h = 0, 1, 2, \ldots$ on p_t is given by

$$\frac{\partial p_t}{\partial d_{t+h}} = (1+r)^{-h-1} \qquad h = 0, 1, \ldots$$

Thus, the effect diminishes the further the change takes place in the future. Alternatively, consider a *permanent* change in dividends, i.e. a change where all dividends increase by some constant amount $\triangle d$. The corresponding price change $\triangle p_t$ equals:

$$\triangle p_t = (1+r)^{-1} \sum_{j=0}^{\infty} (1+r)^{-j} \triangle d = \frac{\triangle d}{r}$$

Similarly a proportional increase of all dividends would lead to the same proportional increase in the share price. It also shows that relatively small permanent changes in the dividends can lead to large fluctuations in the share price. These "comparative" exercises demonstrate that the rational expectations solution which eliminates the bubble term makes sense.

Cagan's Model of Hyperinflation

In periods of hyperinflation the price level rises by more than 50 percent a month. As these periods are usually rather short lived, they can serve as a laboratory for the study of the relation between money supply and the price level because other factors like changes in real output can be disregarded. The model also serves to illustrate the implications of alternative expectations mechanisms, in particular the difference between adaptive and rational expectations. Denoting by m_t the logarithm of the money stock in period t and by p_t the logarithm of the price level in period t, the model first proposed

and analyzed by Cagan (1956) consists of the following three equations.[8]

$$\begin{aligned} m_t^d - p_t &= \alpha(p_{t+1}^e - p_t), \quad \alpha < 0 && \text{(money demand)} \\ m_t^s &= m_t^d = m_t && \text{(money supply)} \\ p_{t+1}^e - p_t &= \gamma(p_t - p_{t-1}), \quad \gamma > 0 && \text{(adaptive expectations)} \end{aligned}$$

The first equation is a money demand equation in logarithmic form. It relates the logged demand for the real money stock, $m_t^d - p_t$, where the superscript d stands for demand, to the rate of inflation expected to prevail in period $t+1$, $p_{t+1}^e - p_t$, where the superscript e stands for expectation. This relation is negative because households and firms want to hold less money if they expect the real value of money to deteriorate in the next period due to high inflation rates. Thus, $\alpha < 0$. In this model, the central bank perfectly controls the money stock and sets it independently of the development of the price level. The model treats the logarithm of the supply of the money stock, m_t^s, where the superscript s stands for supply, as exogenous. The money stock injected in the economy is completely absorbed by the economy so that in each point in time the supply of money equals the demand of money. Combining the first two equations, i.e. replacing m_t^d by m_t in the first equation, leads to a portfolio equilibrium condition. As we will see, the behavior of the model depends crucially on the way in which expectations are formed. Following the original contribution by Cagan, we assume that expectations are formed adaptively, i.e. agents form their expectations by extrapolating past inflation. The third equation postulates a very simple adaptive expectation formation scheme: inflation expected to prevail in the next period is just proportional to the current inflation. Thereby the proportionality factor γ is assumed to be positive, meaning that expected inflation increases if current inflation increases. Combining all three equations of the model and solving for p_t, we arrive at the following linear nonhomogeneous first order difference equation:

$$p_t = \frac{\alpha\gamma}{1+\alpha\gamma} p_{t-1} + \frac{1}{1+\alpha\gamma} m_t = \phi p_{t-1} + z_t \tag{2.24}$$

where $\phi = \frac{\alpha\gamma}{1+\alpha\gamma}$ and $z_t = \frac{1}{1+\alpha\gamma} m_t$.

From our previous discussion we know that the general solution of this difference equation is given as the sum of the general solution to the homogenous equation and a particular solution, $p_t^{(p)}$, to the nonhomogeneous equation:

$$p_t = \phi^t c + p_t^{(p)}$$

[8] See also the analysis in Sargent (1987).

One particular solution can be found by recursively inserting into equation (2.24):

$$\begin{aligned} p_1 &= \phi p_0 + z_1 \\ p_2 &= \phi p_1 + z_2 = \phi^2 p_0 + \phi z_1 + z_2 \\ &\dots \\ p_t &= \phi^t p_0 + \phi^{t-1} z_1 + \phi^{t-2} z_2 + \cdots + \phi z_{t-1} + z_t \\ &= \phi^t p_0 + \sum_{i=0}^{t-1} \phi^i z_{t-i} \end{aligned}$$

This is again an illustration of the superposition principle. The logged price in period t, p_t, is just the sum of two components. The first one is a function of p_0 whereas the second one is a weighted sum of *past* logged money stocks. In economics there is no natural starting period so that one may iterate the above equation further, thereby going back into infinite remote past:

$$p_t = \lim_{i \to \infty} \phi^i p_{t-i} + \sum_{i=0}^{\infty} \phi^i z_{t-i}$$

From a mathematical point of view this expression only makes sense if the limit of the infinite sum exists. Thus, additional assumptions are required. Suppose that logged money remained constant, i.e. $m_t = m < \infty$ for all t, then the logic of the model suggests that the logged price level should remain finite as well. In mathematical terms this means that $\sum_{i \to \infty}^{\infty} \phi^i$ should converge. This is, however, a geometric sum so that convergence is achieved if and only if

$$|\phi| = \left| \frac{\alpha\gamma}{1 + \alpha\gamma} \right| < 1. \tag{2.25}$$

Assuming that this *stability condition* holds, the general solution of the difference equation (2.24) implied by the Cagan model is:

$$p_t = \phi^t c + \sum_{i=0}^{\infty} \phi^i z_{t-i} \tag{2.26}$$

where the constant c can be computed from an initial value condition.[9] Such an initial condition arises naturally because the formation of adaptive expectations requires the knowledge of the price from the previous period which can then serve as an initial condition.

[9]Note that the model assumptions $\alpha < 0$ and $\gamma > 0$ are not sufficient to guarantee the stability condition.

The stability condition therefore has important consequences. First, irrespective of the value of c, the first term of the solution (the general solution to the homogenous equation), $\phi^t c$, becomes less and less important as time unfolds. Thus, for a large enough t, the logged price level will be dominated by the particular solution to the nonhomogeneous equation, $\sum_{i=0}^{\infty} \phi^i z_{t-i}$. In this infinite sum, the more recent values of the money stock are more important for the determination of the price level. The importance of past money stocks diminishes as one goes further back into the past. Third, suppose that money stock is increased by a constant percentage point, Δm, in every period, then the effect on the logged price level, Δp_t is given by

$$\Delta p_t = \sum_{i=0}^{\infty} \phi^i \left(\frac{1}{1+\alpha\gamma} \Delta m \right) = \frac{1}{1-\phi} \ \frac{1}{1+\alpha\gamma} \Delta m = \Delta m.$$

Thus, the price level moves up by the same percentage point. Such a once-and-for-all change is termed a *permanent change*. In contrast a *transitory change* is a change which occurs only once. The effect of a transitory change of m_t by Δm in period t on the logged price level in period $t+h$ for some $h \geq 0$ is given by

$$\Delta p_{t+h} = \phi^h \ \frac{1}{1+\alpha\gamma} \Delta m = \left(\frac{\alpha\gamma}{1+\alpha\gamma} \right)^h \frac{1}{1+\alpha\gamma} \Delta m$$

The values $\frac{\Delta p_{t+h}}{\Delta m}$ seen as a function of $h \geq 0$ are called the *impulse response function*. It gives the reaction of the logged price level over time to a transitory change of the logged money stock. As is clear from the above formula, the stability condition implies that the effect on the logged price level dies out exponentially over time. Usually the impulse response function is plotted as a function of h as in Figure 2.3.

The character of the model changes drastically if *rational expectations* are assumed instead of adaptive expectations. In the context of a deterministic model this amounts to assuming perfect foresight. Thus, the third equation of the model is replaced by

$$p^e_{t+1} = p_{t+1} \tag{2.27}$$

With this change the new difference equation becomes:

$$p_{t+1} = \frac{\alpha - 1}{\alpha} p_t + \frac{m_t}{\alpha} = \phi p_t + z_t \tag{2.28}$$

with $z_t = m_t/\alpha$. As $\phi = \frac{\alpha-1}{\alpha} > 1$, the stability condition is violated. One can nevertheless find a meaningful particular solution of the nonhomogeneous

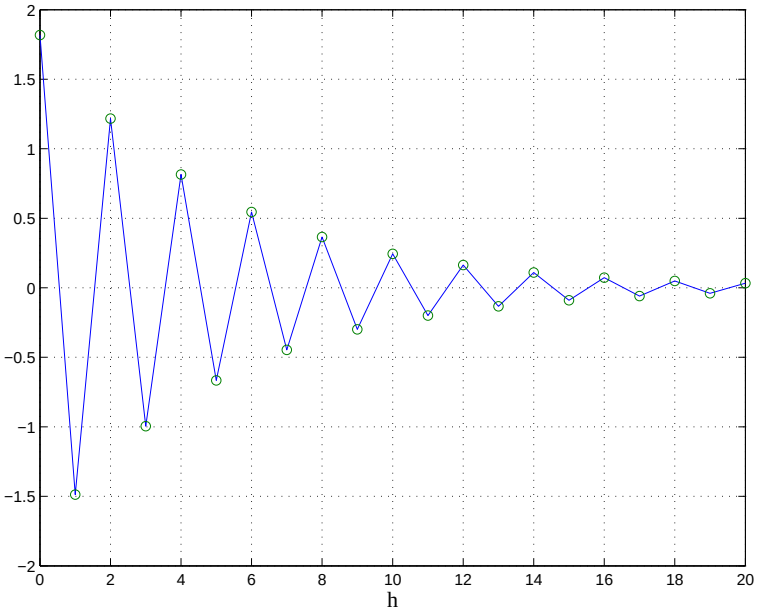


Figure 2.3: Impulse response function of the Cagan model with adaptive expectations taking $\alpha = -0.5$ and $\gamma = 0.9$

equation by iterating the difference equation forwards in time instead of backwards:

$$\begin{aligned} p_t &= \phi^{-1} p_{t+1} - \phi^{-1} z_t \\ &= \phi^{-1}\left(\phi^{-1} p_{t+2} - \phi^{-1} z_{t+1}\right) - \phi^{-1} z_t = \phi^{-2} p_{t+2} - \phi^{-2} z_{t+1} - \phi^{-1} z_t \\ &\dots \\ &= \phi^{-h} p_{t+h} - \phi^{-1} \sum_{i=0}^{h-1} \phi^{-i} z_{t+i} \qquad \text{for } h > 0 \end{aligned}$$

The logged price level in period t, p_t, depends on the expected logged price level in the future, p_{t+h}, and on the development of logged money expected to be realized in the *future*. Because the economy is expected to live forever, this forward iteration is carried on into the infinite future to yield:

$$p_t = \lim_{h\to\infty} \phi^{-h} p_{t+h} - \phi^{-1} \sum_{i=0}^{\infty} \phi^{-i} z_{t+i}$$

As $0 < \phi^{-1} < 1$, the limit and the infinite sum are well defined, provided that the logged money stock remains bounded. Under the assumption that the logged money stock is expected to remain bounded, the economic logic

of the model suggests that the logged price level should remain bounded as well. This suggests the following particular solution to the nonhomogeneous equation:

$$p_t^{(p)} = -\phi^{-1} \sum_{i=0}^{\infty} \phi^{-i} z_{t+i}$$

by the superposition principle the general solution of the nonhomogeneous difference equation (2.28) is:

$$p_t = \phi^t c + p_t^{(p)} = \phi^t c - \phi^{-1} \sum_{i=0}^{\infty} \phi^{-i} z_{t+i} \tag{2.29}$$

Due to the term $\phi^t c$, the logged price level grows exponentially without bound although the logged money stock may be expected to remain bounded, unless $c = 0$. Thus, setting $c = 0$ or equivalently $p_0 = p_0^{(p)}$ guarantees a nonexplosive rational expectations equilibrium.

To summarize, the Cagan model suggests the following two solutions:

$$p_t = \phi^t c_b + p_t^{(b)}, \qquad \text{whereby } p_t^{(b)} = \sum_{i=0}^{\infty} \phi^i z_{t-i}$$

$$p_t = \phi^t c_f + p_t^{(f)}, \qquad \text{whereby } p_t^{(f)} = -\phi^{-1} \sum_{i=0}^{\infty} \phi^{-i} z_{t+i}$$

Which of the two solutions is appropriate depends on the value of ϕ. If $|\phi| < 1$ only the first solution delivers sensible paths for p_t, i.e. paths which do not explode for bounded values of z_t. However, we have a whole family of paths parameterized by the constant c_b. Only when we chose a particular initial value for p_{t_0} for some t_0, or equivalently a value for c_b, will the price path be uniquely determined. In the case $|\phi| > 1$ which is implied by the assumption of rational expectations, only the second solution is meaningful because it delivers a well-defined particular solution for bounded z_t's. However, the general solution to the homogenous equation implies an exploding price level except for $c_f = 0$. Thus, there is only one non-exploding solution in this case: $p_t = p_t^{(f)}$. The price level therefore equals in each period its steady state level. The assumption of rational expectations together with the assumption that a bounded forcing variable should lead to a bounded price path pinned down a unique solution. Thus, the price level is *determined* without the need of an initial condition.

2.4 Difference Equations of Order p

We next turn to the general case represented by equation (2.1). As can be easily verified if $x_t^{(1)}$ and $x_t^{(2)}$ are two particular solutions of the nonhomogeneous equation, $x_t^{(1)} - x_t^{(2)}$ is a solution to the homogeneous equation:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p}, \qquad \phi_p \neq 0. \tag{2.30}$$

Thus, the superposition principle stated in Theorem 2.2 also holds in the general case: the general solution to the nonhomogeneous equations can be represented as the sum of the general solution to the homogeneous and a particular solution to the nonhomogeneous equation. Thus, we begin the analysis of the general case by an investigation of the homogeneous equation.

2.4.1 Homogeneous Difference Equation of Order p

In order to find the general solution of the homogeneous equation, we guess that it will be of the same form as in the first order case, i.e. of the form $\lambda^t c, c \neq 0$. Inserting this guess into the homogeneous equation (2.30), we get:

$$\lambda^t c = \phi_1 \lambda^{t-1} c + \phi_2 \lambda^{t-2} c + \cdots + \phi_p \lambda^{t-p} c$$

which after cancelling out c, dividing by λ^t and substituting z for $\frac{1}{\lambda}$ leads to:

$$1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0 \tag{2.31}$$

This equation is called the *characteristic equation* of the homogeneous equation (2.30). Thus, in order for $\lambda^t x$ to be a solution to the homogeneous equation $z = \frac{1}{\lambda}$ must be a root to the characteristic equation (2.31). These roots are called the *characteristic roots*. Note that the assumption $\phi_p \neq 0$ implies that none of the characteristic roots is equal to zero.

From the Fundamental Theorem of Algebra we know that there are p, possibly complex, roots to the characteristic equation. Denote these roots by $z_1, \ldots, z_p$ and their corresponding $\lambda's$ by $\lambda_1, \ldots, \lambda_p$. To facilitate the discussion consider first the standard case where all p roots are distinct.

distinct roots

In this case we have the following theorem.

Theorem 2.3 (Fundamental Set for equation of order p)**.** If all the roots of the characteristic equation are distinct, the set $\{\lambda_1^t, \ldots, \lambda_p^t\}$ forms a fundamental set of solutions.

Proof. It suffices to show that $\det \mathcal{C}(t) \neq 0$ where $\mathcal{C}(t)$ is the Casarotian matrix of $\{\lambda_1^t, \ldots, \lambda_p^t\}$.

$$\begin{aligned}\det \mathcal{C}(t) &= \det \begin{pmatrix} \lambda_1^t & \lambda_2^t & \cdots & \lambda_p^t \\ \lambda_1^{t+1} & \lambda_2^{t+1} & \cdots & \lambda_p^{t+1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{t+p-1} & \lambda_2^{t+p-1} & \cdots & \lambda_p^{t+p-1} \end{pmatrix} \\ &= \lambda_1^t \lambda_2^t \ldots \lambda_p^t \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_p \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{p-1} & \lambda_2^{p-1} & \cdots & \lambda_p^{p-1} \end{pmatrix}\end{aligned}$$

This second matrix is called the Vandermonde matrix whose determinant equals $\prod_{1 \leq i < j \leq p} (\lambda_j - \lambda_i)$ which is different from zero because the roots are distinct. Thus, $\det \mathcal{C}(t) \neq 0$, because the roots are also different from zero. □

The above Theorem thus implies that the general solution to the homogeneous equation $x_t^{(g)}$ is given by

$$x_t^{(g)} = \lambda_1^t c_1 + \lambda_2^t c_2 + \cdots + \lambda_p^t c_p \tag{2.32}$$

where $c_1, \ldots, c_p$ are given constants yet to be determined from boundary conditions. Using the same technique as in the proof of Theorem 2.1, it is easy to demonstrate that the set of solutions forms a linear space of dimension p.

multiple roots

When the roots of the characteristic equation are not distinct, the situation becomes more complicated. Denote the r distinct roots by $z_1, \cdots, z_r$, $r < p$, and their corresponding multiplicities by $m_1, \cdots, m_r$. Writing the homogeneous difference equation in terms of the lag operator leads to

$$\begin{aligned}&(1 - \phi_1 \mathrm{L} - \cdots - \phi_p \mathrm{L}^p)\, x_t \\ &\qquad = (1 - \lambda_1 \mathrm{L})^{m_1} (1 - \lambda_2 \mathrm{L})^{m_2} \cdots (1 - \lambda_r \mathrm{L})^{m_r}\, x_t = 0\end{aligned} \tag{2.33}$$

where L denotes the lag operator[10] and λ_i, $1 \leq i \leq r$, equals $\frac{1}{z_i}$. In order to find the general solution, we will proceed in several steps. First note if ψ_t is a solution to

$$(1 - \lambda_i \mathrm{L})^{m_i}\, \psi_t = 0 \tag{2.34}$$

[10]The lag or shift operator transforms a sequence $\{x_t\}$ into $\{x_{t-1}\}$, i.e. $\mathrm{L}x_t = x_{t-1}$.

it is also a solution to (2.33). Second, $\mathcal{G}_i = \{\lambda_i^t, t\lambda_i^t, t^2\lambda_i^t, \cdots, t^{m_i-1}\lambda_i^t\}$ is a fundamental set of solutions for equation (2.34). Before we prove this statement in Lemma 2.4, we need the following lemma.

Lemma 2.3. For all $k \geq 1$

$$(1 - \mathrm{L})^k t^s = 0, \qquad 0 \leq s < k$$

Proof. The application of the operator $1 - \mathrm{L}$ on t^s leads to a polynomial of degree $s-1$, because the term t^s cancels in $(1-\mathrm{L})t^s = t^s - (t-1)^s$ and only terms of degree smaller than s remain. Applying $1 - \mathrm{L}$ again reduces the degree of the polynomial again by one. Finally, $(1-\mathrm{L})^s t^s$ leads to a constant. Thus, $(1 - \mathrm{L})^{s+1} t^s = 0$. This proves the lemma because further applications of $1 - L$ will again result in zero. □

Lemma 2.4. The set $\mathcal{G}_i = \{\lambda_i^t, t\lambda_i^t, t^2\lambda_i^t, \cdots, t^{m_i-1}\lambda_i^t\}$ represents a fundamental set of solutions to the equation (2.34).

Proof. Take s, $1 \leq s \leq m_i - 1$, then

$$(1 - \lambda_i \mathrm{L})^{m_i} (t^s \lambda_i^t) = \lambda_i^t (1 - \mathrm{L})^{m_i} (t^s) = 0$$

because $(1-\mathrm{L})^{m_i} t^s = 0$ according to Lemma 2.3.[11] Therefore $t^s\lambda_i^t$ is a solution to (2.34). The set $\mathcal{G}_i$ is linearly independent because the set $\{1, t, t^2, \cdots, t^{m_i-1}\}$ is linearly independent. □

It is then easily seen that $\mathcal{G} = \bigcup_{i=1}^r \mathcal{G}_i$ is a fundamental set of solutions to the equation (2.33). Thus, the general solution can be written as

$$x_t = \sum_{i=1}^{r} \left(c_{i0} + c_{i1}t + c_{i2}t^2 + \cdots + c_{i,m_i-1}t^{m_i-1}\right) \lambda_i^t \tag{2.35}$$

where $c_{i0}, \ldots, c_{i,m_i}$, $i = 1, \ldots, r$, are given constants yet to be determined from boundary conditions. As before, the set of solutions forms a linear space of order p because $\sum_{i=1}^r m_i = p$.

2.4.2 Nonhomogeneous Equation of Order p

As in the case of homogeneous difference equations of order one, the set of all solutions forms a linear space. The dimension of this space is given by the order of the difference equation, i.e. by p. Consider two solutions,

[11] Here we made use of the relation $P(\mathrm{L})(\lambda^t g(t)) = \lambda^t P((\lambda^{-1}\mathrm{L}))g(t)$ where $P(\mathrm{L})$ is a lag polynomial and g is any discrete function.

$x^{(1)}$ and $x^{(2)}$, of the nonhomogeneous equation. It is easy to verify that $x^{(1)} - x^{(2)}$ is then a solution to the homogeneous equation. This implies that the superposition principle also applies to nonhomogeneous equations of order p greater than one. Thus the general solution of the nonhomogeneous equation can be written as before as

$$x_t = x_t^{(g)} + x_t^{(p)}$$

where $x_t^{(g)}$ is the general solution to the homogeneous equation given by equation (2.35) and $x_t^{(p)}$ is a particular solution to the nonhomogeneous equation.

In the search for a particular solution, the same ideas as in first order case can be used. If the nonhomogeneous part is constant, i.e. $z_t = z$, the steady state, if it exists, qualifies for a particular solution to the nonhomogeneous equation. If the nonhomogeneous part depends on time, a particular solution can be found by iterating the equation backwards and/or forwards depending on the location of the roots. This will become clear by analyzing the examples in section 2.4.4.

2.4.3 Limiting Behavior of Solutions

Before turning to some economic examples, we classify to the qualitative behavior of the solutions. In particular, we will explore the stability properties of the steady states and the limiting behavior of the solutions, i.e. the behavior when time goes to infinity. The analysis can be reduced to the discussion of second order homogenous equations:

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} = 0, \qquad \phi_2 \neq 0. \tag{2.36}$$

Higher order equations will add no new qualitative features. Assuming that $1-\phi_1-\phi_2 \neq 0$, the unique fixed point of this homogenous equation is 0. The corresponding characteristic equation is given by the quadratic equation:

$$1 - \phi_1 z - \phi_2 z^2 = 0.$$

The solutions of this equation are given by the familiar formula:

$$z_{1,2} = -\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}.$$

Or in terms of $\lambda = \frac{1}{z}$:

$$\lambda_{1,2} = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}. \tag{2.37}$$

To understand the qualitative behavior of x_t, we distinguish three cases:

λ_1 and λ_2 are real and distinct: The general solution is given by

$$x_t = \lambda_1^t c_1 + \lambda_2^t c_2 = \lambda_1^t \left[c_1 + \left(\frac{\lambda_2}{\lambda_1} \right)^t c_2 \right]$$

Suppose without loss of generality that $|\lambda_1| > |\lambda_2|$ so that $\left(\frac{\lambda_2}{\lambda_1}\right)^t \to 0$ as $t \to \infty$. This implies that the behavior of x_t is asymptotically governed by the larger root λ_1:

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} \lambda_1^t c_1$$

Depending on the value of λ_1, six different cases emerge:

1. $\lambda_1 > 1$: $\lambda_1^t c_1$ diverges to ∞ as $t \to \infty$. The fixed point zero is unstable.
2. $\lambda_1 = 1$: $\lambda_1^t c_1$ remains constant and x_t approaches c_1 asymptotically. Starting the system with initial conditions $x_1 = x_0 = x$, x arbitrary, which is equivalent to $c_1 = x$ and $c_2 = 0$, x_t will remain at this value x forever.
3. $0 < \lambda_1 < 1$: $\lambda_1^t c_1$ decreases monotonically to zero. Zero is an asymptotically stable fixed point.
4. $-1 < \lambda_1 < 0$: $\lambda_1^t c_1$ oscillates around zero, alternating in sign, but converges to zero. Zero is again an asymptotically stable fixed point.
5. $\lambda_1 = -1$: $\lambda_1^t c_1$ alternates between the values c_1 and $-c_1$. Thus, the sequence (x_t) will have two accumulation points c_1 and $-c_1$.
6. $\lambda_1 < -1$: $c_1 \lambda_1^t$ alternates in sign, but diverges in absolute value to ∞. The fixed point zero is unstable.

The behavior of x_t in all six cases is illustrated in Figure 2.4.

equal roots $\lambda = \lambda_1 = \lambda_2$: According to (2.35) the solution is given by: $x_t = (c_1 + c_2 t)\,\lambda^t$. Clearly, if $\lambda \geq 1$, x_t diverges monotonically; or, if $\lambda \leq -1$, x_t diverges alternating signs. For $|\lambda| < 1$, the solution converges to zero, because $\lim_{t\to\infty} t\lambda^t = 0$.

complex roots: The two roots appear as complex conjugate pairs and may be written as $\lambda_1 = \alpha + \imath\beta$ and $\lambda_2 = \alpha - \imath\beta$ with $\beta \neq 0$. In terms of polar coordinates the two roots may alternatively be written as $\lambda_1 = re^{\imath\theta}$,

respectively $\lambda_2 = re^{-\imath\theta}$, where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$. The solution is then given by

$$\begin{aligned} x_t &= \lambda_1^t c_1 + \lambda_2^t c_2 = (\alpha + \imath\beta)^t c_1 + (\alpha - \imath\beta)^t c_2 \\ &= r^t e^{\imath\theta t} c_1 + r^t e^{-\imath\theta t} c_2 \\ &= r^t \left[c_1 \left(\cos(\theta t) + \imath \sin(\theta t)\right) + c_2 \left(\cos(\theta t) - \imath \sin(\theta t)\right)\right] \\ &= r^t \left[(c_1 + c_2)\cos(\theta t) + \imath(c_1 - c_2)\sin(\theta t)\right] \end{aligned}$$

Since x_t must be a real number, $c_1 + c_2$ must also be real whereas $c_1 - c_2$ must be purely imaginary. This implies that c_1 and c_2 must be complex conjugate. In terms of polar coordinates they can be written as $c_1 = \rho e^{\imath\omega}$ and $c_2 = \rho e^{-\imath\omega}$ for some ρ and some ω. Inserting into the above equation finally gives:

$$\begin{aligned} x_t &= \rho r^t \left[e^{\imath(\theta t + \omega)} + e^{-\imath(\theta t + \omega)}\right] \\ &= 2\rho r^t \cos(\theta t + \omega) \end{aligned}$$

The solution therefore clearly oscillates because the cosine function oscillates. Depending on the location of the conjugate roots three cases must be distinguished:

1. $r > 1$: both roots are outside the unit circle (i.e. the circle of radius one and centered in the point $(0, 0)$). x_t oscillates, but with ever increasing amplitude. The fixed point zero is unstable.
2. $r = 1$: both roots are on the unit circle. x_t oscillates, but with constant amplitude.
3. $r < 1$: both roots are inside the unit circle. The solution oscillates, but with monotonically decreasing amplitude and converges to zero as $t \to \infty$. The fixed point zero is asymptotically stable.

Figure 2.5 illustrates the three cases.

We can summarize the above discussion in the following theorem.

Theorem 2.4 (Limiting Behavior of Second Order Equation)**.** The following statements hold in the case of linear homogenous difference equation of order two (equation (2.36)):

(i) All solutions oscillate around zero if and only if the equation has no positive real characteristic root.

(ii) All solutions converge to zero (i.e. zero is an asymptotically stable steady state) if and only if $\max\{|\lambda_1|, |\lambda_2|\} < 1$.

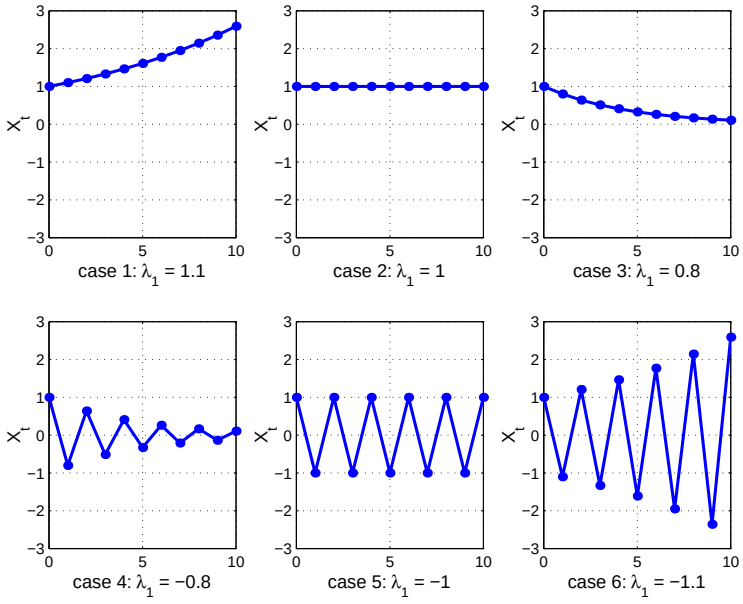


Figure 2.4: Behavior of $x_t = \lambda_1^t$ depending on $\lambda \in \mathbb{R}$

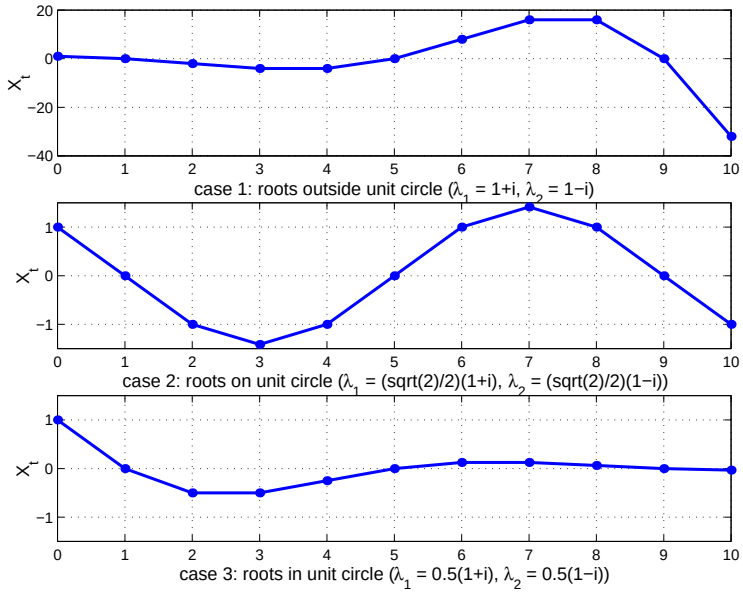


Figure 2.5: Behavior of x_t in case of complex roots

Although the limiting behavior of x_t is most easily understood in terms of the roots of the characteristic equation, it is sometimes more convenient to analyze the properties of the difference equation in terms of the original parameters ϕ_1 and ϕ_2. Consider for this purpose a nonhomogeneous second order difference equation where the nonhomogeneous part is just a constant equal to Z:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + z, \qquad z \neq 0. \tag{2.38}$$

Zero is no longer an equilibrium point. Instead, the new equilibrium point x^* can be found by solving the equation:

$$x^* = \phi_1 x^* + \phi_2 x^* + z \quad \Rightarrow \quad x^* = \frac{z}{1 - \phi_1 - \phi_2}$$

Note that an equilibrium only exists if $1 - \phi_1 - \phi_2 \neq 0$. This condition is equivalent to the condition that 1 cannot be a root. As the steady state qualifies for a particular solution of equation (2.38), the general solution is given by

$$x_t = x^* + x_t^{(g)}$$

Thus, x_t converges to its equilibrium if and only if $x_t^{(g)}$ converges to zero as $t \to \infty$. Moreover, the solution oscillates around x^* if and only if $x_t^{(g)}$ oscillates around zero. Based on the theorem just above, the following theorems hold.

Theorem 2.5 (Limiting Behavior of Second Order Equation (original parameters))**.** Assuming $1 - \phi_1 - \phi_2 \neq 0$, the following statements hold.

(i) All solutions of the nonhomogeneous equation (2.38) oscillate around the equilibrium point x^* if and only if the characteristic equation has no positive real characteristic root.

(ii) All solutions to the nonhomogeneous equation (2.38) converge to x^* (i.e. x^* is asymptotically stable) if and only if $\max\{|\lambda_1|, |\lambda_2|\} < 1$.

Theorem 2.6 (Stability Conditions of Second Order Equation (original parameters))**.** The equilibrium point x^* is asymptotically stable (i.e. all solutions converge to x^*) if and only if the following three conditions are satisfied:

(i) $1 - \phi_1 - \phi_2 > 0$

(ii) $1 + \phi_1 - \phi_2 > 0$

(iii) $1 + \phi_2 > 0$

Proof. Assume that x^* is an asymptotically stable equilibrium point. According to the previous Theorem (2.4), this means that both λ_1 and λ_2 must be smaller than one in absolute value. According to equation (2.37) this implies that

$$|\lambda_1| = \left| \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \right| < 1 \quad \text{and} \quad |\lambda_2| = \left| \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2} \right| < 1 \, .$$

Two cases have to be distinguished.

real roots: $\phi_1^2 + 4\phi_2 > 0$***:*** This implies the set of inequalities:

$$-2 < \phi_1 + \sqrt{\phi_1^2 + 4\phi_2} < 2$$
$$-2 < \phi_1 - \sqrt{\phi_1^2 + 4\phi_2} < 2$$

or, equivalently,

$$-2 - \phi_1 < \sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1$$
$$-2 - \phi_1 < -\sqrt{\phi_1^2 + 4\phi_2} < 2 - \phi_1$$

Squaring the second inequality in the first line implies: $\phi_1^2 + 4\phi_2 < 4 - 4\phi_1 + \phi_1^2$ which leads to condition (i). Similarly, squaring the first inequality in the second line yields: $4 + 4\phi_1 + \phi_1^2 > \phi_1^2 + 4\phi_2$ which results in condition (ii). The assumption $|\lambda_1| < 1$ and $|\lambda_2| < 1$ imply that $|\lambda_1 \lambda_2| = |-\phi_2| < 1$ which gives condition (iii).

complex roots: $\phi_1^2 + 4\phi_2 < 0$***:*** This implies that $0 < \phi_1^2 < -4\phi_2$. Therefore

$$4(1 - \phi_1 - \phi_2) > 4 - 4\phi_1 + \phi_1^2 = (2 - \phi_1)^2 > 0$$

which is equivalent to condition (i). Similarly,

$$4(1 + \phi_1 - \phi_2) > 4 + 4\phi_1 + \phi_1^2 = (2 + \phi_1)^2 > 0$$

which is equivalent to condition (ii). In order to obtain condition (iii), note that the two complex conjugate roots are given by

$$\lambda_1 = \frac{\phi_1}{2} + \frac{\imath}{2}\sqrt{\phi_1^2 + 4\phi_2} \quad \text{and} \quad \lambda_2 = \frac{\phi_1}{2} - \frac{\imath}{2}\sqrt{\phi_1^2 + 4\phi_2}.$$

Because $|\lambda_1| < 1$ and $|\lambda_2| < 1$ by assumption, we have that $|\lambda_1 \lambda_2| = |-\phi_2| < 1$ which is condition (iii).

Assume now that the three conditions are satisfied. They immediately imply that $-2 < \phi_1 < 2$ and that $-1 < \phi_2 < 1$. If the roots are real then

$$\begin{aligned} -1 < \frac{-2+\sqrt{\phi_1^2+4\phi_2}}{2} &< \lambda_1 = \frac{\phi_1+\sqrt{\phi_1^2+4\phi_2}}{2} \\ &< \frac{\phi_1+\sqrt{\phi_1^2+4-4\phi_1}}{2} \\ &= \frac{\phi_1+\sqrt{(2-\phi_1)^2}}{2} \\ &= \frac{\phi_1-\phi_1+2}{2} = 1 \end{aligned}$$

Similarly,

$$\begin{aligned} 1 > \frac{2-\sqrt{\phi_1^2+4\phi_2}}{2} &> \lambda_2 = \frac{\phi_1-\sqrt{\phi_1^2+4\phi_2}}{2} \\ &> \frac{\phi_1-\sqrt{\phi_1^2+4+4\phi_1}}{2} \\ &= \frac{\phi_1-\sqrt{(\phi_1+2)^2}}{2} \\ &= \frac{\phi_1-\phi_1-2}{2} = -1 \end{aligned}$$

If the roots are complex, λ_1 and λ_2 are complex conjugate numbers. Their squared modulus then equals $\lambda_1\lambda_2 = \frac{\phi_1^2-(\phi_1^2+4\phi_2)}{4} = -\phi_2$. As $-\phi_2 < 1$, the modulus of both λ_1 and λ_2 is smaller than one. □

The three conditions listed above determine a triangle in the ϕ_1-ϕ_2-plane with vertices $(-2,-1)$, $(0,1)$ and $(2,-1)$. Points inside the triangle imply an asymptotically stable behavior whereas points outside the triangle lead to an unstable behavior. The parabola $\phi_1^2 + 4\phi_2 = 0$ determines the region of complex roots. Values of ϕ_1 and ϕ_2 above the parabola lead to real roots whereas values below the parabola lead to complex roots. The situation is represented in Figure 2.6.

2.4.4 Examples

Multiplier Accelerator model

A classic economic example of a second order difference equation is the multiplier-accelerator model originally proposed by Samuelson (1939). It

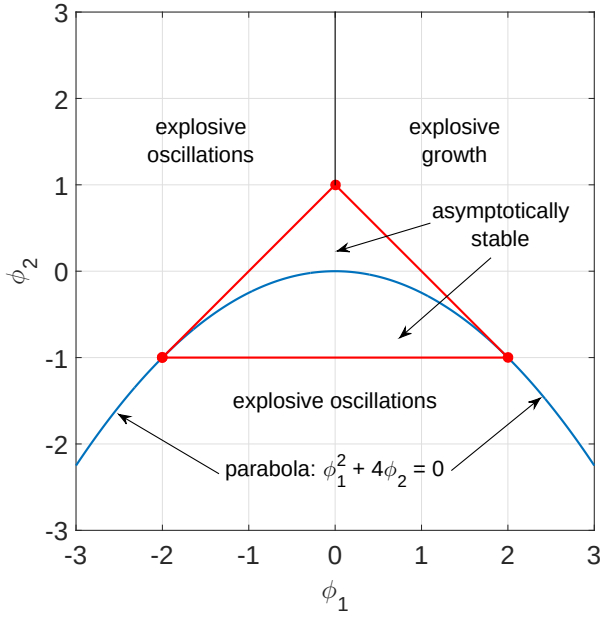


Figure 2.6: Stability properties of equation: $x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} = 0$

was designed to demonstrate how the interaction of the multiplier and the accelerator can generate business cycles. The model is one of a closed economy and consists of a consumption function, an investment function which incorporates the accelerator idea and the income identity:

$$\begin{aligned} C_t &= \alpha + \beta Y_{t-1}, & 0 < \beta < 1, \alpha > 0 \quad & \text{(consumption)} \\ I_t &= \gamma(Y_{t-1} - Y_{t-2}), & \gamma > 0 \quad & \text{(investment)} \\ Y_t &= C_t + I_t + G_t, & & \text{(income identity)} \end{aligned}$$

where C_t, I_t, Y_t, and G_t denotes private consumption expenditures, investment expenditures, income, and government consumption, respectively. The parameter β is called the marginal propensity of consumption and is assumed to be between zero and one. The remaining parameters of the model, α and γ, bear no restriction besides that they have to be positive. Inserting the consumption and the investment equation into the income identity leads to the following nonhomogeneous second order difference equation:

$$Y_t = (\beta + \gamma)Y_{t-1} - \gamma Y_{t-2} + (\alpha + G_t) \tag{2.39}$$

If government expenditures remain constant over time and equal to G, the equilibrium point Y^* for equation (2.39) can be computed as follows:

$$Y^* = (\beta + \gamma)Y^* - \gamma Y^* + \alpha + G \quad \Rightarrow \quad Y^* = \frac{\alpha + G}{1 - \beta}$$

The stability of this equilibrium point can be investigated by verifying if the three conditions of Theorem 2.6 are satisfied:

(i) $1 - (\beta + \gamma) + \gamma = 1 - \beta > 0$

(ii) $1 + (\beta + \gamma) + \gamma = 1 + \beta + 2\gamma > 0$

(iii) $1 - \gamma > 0$

Given the assumptions of the model, the first two conditions are automatically satisfied. The third condition, however, is only valid if the accelerator is not too strong, i.e. if $\gamma < 1$. The steady state Y^* is therefore asymptotically stable if one imposes this additional requirement. Y_t oscillates around its steady state if, according to Theorem 2.4, there is no real positive inverse root of the characteristic equation. The inverse of the characteristic roots are given by

$$\lambda_{1,2} = \frac{(\beta + \gamma) \pm \sqrt{(\beta + \gamma)^2 - 4\gamma}}{2}.$$

If the roots are real, they are both strictly positive and strictly smaller than one. Thus, Y_t can only oscillate around its steady state if and only if the roots are complex, i.e. if $(\beta+\gamma)^2 - 4\gamma < 0$. If they are complex, their moduli are strictly smaller than one.

In the general case where government expenditures are not constant, but vary over time, we apply the method of *undetermined coefficients* to find a particular solution, $Y_t^{(p)}$, to equation (2.39). This method conjectures a certain type of solution and then tries to pin down a solution by inserting it into the difference equation. In the particular case at hand, the roots of the characteristic function are all outside the unit circle. Thus, we conjecture a particular solution of the form:

$$Y_t^{(p)} = c + \sum_{i=0}^{\infty} \psi_i G_{t-i}$$

The coefficients ψ_j are called *impulse responses* or *dynamic multipliers*. They trace the effect on output of an impulse (stimulus) in government expenditures over time. Thereby a unit impulse is specified as $\Delta G_t = 1$ and

$\Delta G_{t-i} = 0$ for $i \neq 0$. The effect on output is then

$$\Delta Y_{t+h}^{(p)} = \sum_{i=0}^{\infty} \psi_i \Delta G_{t+h-i} = \psi_h \Delta G_t = \psi_h, \qquad h = 0, 1, 2, \ldots$$

Inserting this conjectured particular solution into the difference equation leads to

$$\begin{aligned} c + \sum_{i=0}^{\infty} \psi_i G_{t-i} =& c(\beta+\gamma) + (\beta+\gamma) \sum_{i=0}^{\infty} \psi_i G_{t-1-i} \\ & - c\gamma - \gamma \sum_{i=0}^{\infty} \psi_i G_{t-2-i} + \alpha + G_t \end{aligned}$$

Equating the constant terms leads to an equation for c:

$$c\,(1 - (\beta+\gamma) + \gamma) = \alpha \quad \Rightarrow \quad c = \frac{\alpha}{1-\beta} > \alpha > 0$$

Equating the terms for G_{t-i}, $i = 0, 1, \cdots$ leads to:

$$\begin{aligned} \psi_0 &= 1 \\ \psi_1 &= (\beta+\gamma)\psi_0 \quad \Rightarrow \quad \psi_1 = \beta + \gamma \\ \psi_2 &= (\beta+\gamma)\psi_1 - \gamma\psi_0 \\ &\cdots \\ \psi_j &= (\beta+\gamma)\psi_{j-1} - \gamma\psi_{j-2}, \quad j \geq 2 \end{aligned}$$

Thus, the coefficients ψ_j, $j \geq 2$, follow the same homogenous second order difference equation with initial values $\psi_0 = 1$ and $\psi_1 = \beta + \gamma$. The solution can therefore be written as

$$\psi_j = d_1 \lambda_1^j + d_2 \lambda_2^j.$$

The coefficients d_1 and d_2 can then be determined from the initial conditions:

$$\begin{aligned} \psi_0 &= 1 = d_1 + d_2 \\ \psi_1 &= \beta + \gamma = d_1\lambda_1 + d_2\lambda_2 \end{aligned}$$

In order to illustrate the behavior of the multiplier-accelerator model, we discuss several numerical examples.

$\beta = \frac{4}{5}$ **and** $\gamma = \frac{1}{5}$ In this case both roots are real and equal to

$$\lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{5}}{10} = \begin{cases} 0.7236 \\ 0.2764 \end{cases}$$

Therefore the impulse response coefficients ψ_j for $j \geq 0$ are given by $\psi_j = d_1\lambda_1^j + d_2\lambda_2^j$. The constants d_1 and d_2 can be recovered from the initial conditions: $\psi_0 = 1 = d_1 + d_2$ and $\psi_1 = 1 = d_1\lambda_1 + d_2\lambda_2$. Solving these two equations for d_1 and d_2 yields:

$$\begin{aligned} d_1 &= \frac{1-\lambda_2}{\lambda_1-\lambda_2} = 1.6180 \\ d_2 &= \frac{\lambda_1-1}{\lambda_1-\lambda_2} = -0.6180 \end{aligned}$$

The corresponding impulse response function is plotted in Figure 2.7. The initial increase of government expenditures by one unit raises output in current and the subsequent period by one unit. Then the effect of the impulse dies out monotonically. After ten periods the effect almost vanished.

$\beta = \frac{3}{4}$ **and** $\gamma = \frac{1}{4}$ In this case we have a multiple root equal to $\lambda = 0.5$. According to equation (2.35) the impulse response coefficients are therefore given by $\psi_j = (d_0 + d_1 t)\lambda^t$. The constants d_0 and d_1 can again be found by solving the equation system: $\psi_0 = 1 = d_0$ and $\psi_1 = 1 = (d_0 + d_1)\lambda$. The solution is given by $d_0 = 1$ and $d_1 = 1$. The corresponding impulse response coefficients are plotted in Figure 2.7. They resemble very much to those of the previous case. They even die out more rapidly.

$\beta = \frac{2}{3}$ **and** $\gamma = \frac{2}{3}$ In this case the discriminant is negative so that we have two complex conjugate roots:

$$\lambda_{1,2} = \frac{1}{3}\left(2 \pm \imath\sqrt{2}\right)$$

The constants can again be found by solving the equation system: $\psi_0 = 1 = d_1 + d_2$ and $\psi_1 = \frac{4}{3} = d_1\lambda_1 + d_2\lambda_2$. The solution is given by

$$\begin{aligned} d_1 &= \frac{1}{2} - \imath\frac{\sqrt{2}}{2} \\ d_2 &= \frac{1}{2} + \imath\frac{\sqrt{2}}{2} \end{aligned}$$

The corresponding impulse response coefficients are plotted in Figure 2.7. As expected they clearly show an oscillatory behavior. Due to the accelerator, the initial impulse is amplified in period one. The effect is around 1.3. After period one the effect rapidly declines and becomes even negative in period four. However, in period seven the effect starts to increase and becomes again positive in period ten. As is also evident from the Figure, these oscillatory movements die out.

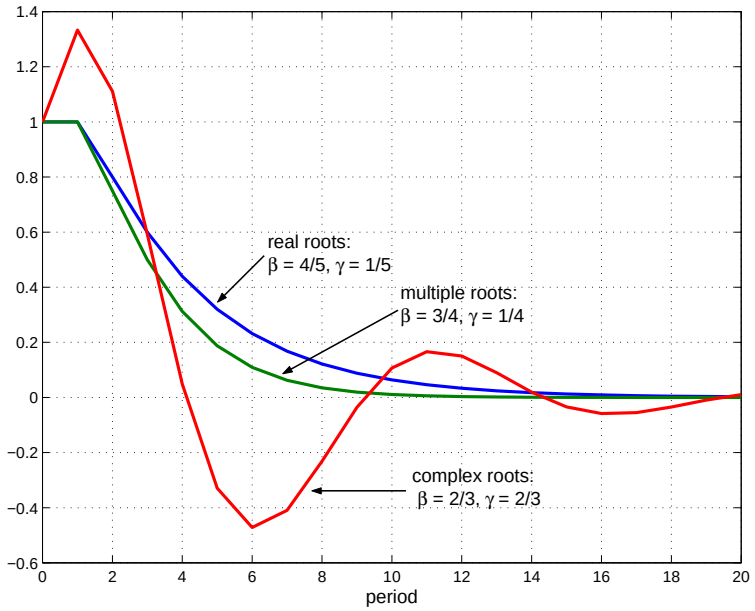


Figure 2.7: Impulse Response Coefficients of the Multiplier-Accelerator model

Cobweb model with Inventory

In this example we extend the simple Cobweb model analyzed in subsection 2.3 by allowing the good to be stored (see Sargent, 1987). In addition, we assume that expectations are rational which in the context of a deterministic model is equivalent to perfect foresight. These extensions will lead to further insights into the method of undetermined coefficients introduced in the previous example. The new set of equations then reads as follows:

$$
\begin{aligned}
D_t &= -\beta p_t, & \beta > 0 \quad & \text{(demand)} \\
S_t &= \gamma p_t^e + u_t, & \gamma > 0 \quad & \text{(supply)} \\
I_t &= \alpha(p_{t+1}^e - p_t), & \alpha > 0 \quad & \text{(inventory demand)} \\
S_t &= D_t + (I_t - I_{t-1}), & & \text{(market clearing)} \\
p_t^e &= p_t, & & \text{(perfect foresight)}
\end{aligned}
$$

where u_t denotes again a supply shock. The inventory demand schedule incorporates a speculative element because inventories will be built up if prices are expected to be higher next period. The market clearing equation shows that the supply which remains unsold is used to build up inventories; on the other hand demand can not only be served by newly supplied goods, but can also be fulfilled out of inventories. Combining these equations leads

to the following second order linear difference equation in the price:

$$p_{t+1} = \frac{\gamma + 2\alpha + \beta}{\alpha} p_t - p_{t-1} + \frac{u_t}{\alpha} \tag{2.40}$$

Setting $\phi = \frac{\gamma+2\alpha+\beta}{\alpha}$, the characteristic equation becomes:

$$1 - \phi z + z^2 = 0$$

This equation implies that the two roots, z_1 and z_2, are given by

$$z_{1,2} = \frac{\phi \pm \sqrt{\phi^2 - 4}}{2}$$

First note that because $\phi > 2$ the roots are real, distinct, and positive. Second they come in reciprocal pairs as $z_1 z_2 = 1$. Thus, one root is smaller than one whereas the other is necessarily greater than one. Thus, we have one stable and one explosive root. The solution to the homogenous equation can therefore be written as

$$p_t = c_1 \lambda^t + c_2 \lambda^{-t}$$

where, without loss of generality, $z_1 = \lambda < 1$ and $z_2 = \frac{1}{\lambda}$. c_1 and c_2 are constants yet to be determined.

Because the agents in this model have rational expectations which implies that they are forward looking, they will incorporate expected future developments of the supply shock into their decision. However, past decision are reflected in the inventories carried over last period. Thus, we conjecture that the solution will have both a forward *and* a backward looking component. Thus, we seek for a particular solution of the following form:

$$p_t = \sum_{j=-\infty}^{\infty} \psi_j u_{t-j}$$

Following the method of undetermined coefficients we insert this guess into the difference equation to get:

$$\sum_{j=-\infty}^{\infty} \psi_j u_{t+1-j} = \phi \sum_{j=-\infty}^{\infty} \psi_j u_{t-j} - \sum_{j=-\infty}^{\infty} \psi_j u_{t-1-j} + \frac{u_t}{\alpha}$$

Writing this equation in extensive form leads to:

$$\begin{aligned}
\cdots \quad + \quad & \psi_{-1}u_{t+2} + \psi_0 u_{t+1} + \psi_1 u_t + \psi_2 u_{t-1} + \psi_3 u_{t-2} + \cdots \\
= \quad & \cdots + \phi\psi_{-2}u_{t+2} + \phi\psi_{-1}u_{t+1} + \phi\psi_0 u_t + \phi\psi_1 u_{t-1} + \phi\psi_2 u_{t-2} + \cdots \\
- \quad & \cdots - \psi_{-3}u_{t+2} - \psi_{-2}u_{t+1} - \psi_{-1}u_t - \psi_0 u_{t-1} - \psi_1 u_{t-2} - \cdots \\
+ \quad & \frac{u_t}{\alpha}
\end{aligned}$$

Equating terms for u_{t-j}, $j = \ldots, -2, -1, 0, 1, 2, \ldots$ gives:

$$\begin{array}{ll} & \ldots \\ u_{t+2}: & \psi_{-1} = \phi\psi_{-2} - \psi_{-3} \\ u_{t+1}: & \psi_0 = \phi\psi_{-1} - \psi_{-2} \\ u_t: & \psi_1 = \phi\psi_0 - \psi_{-1} + \dfrac{1}{\alpha} \\ u_{t-1} & \psi_2 = \phi\psi_1 - \psi_0 \\ u_{t-2}: & \psi_3 = \phi\psi_2 - \psi_1 \\ & \ldots \end{array}$$

This shows that the ψ_j's follow homogenous second order difference equations:

$$\begin{aligned} \psi_j &= \phi\psi_{j-1} - \psi_{j-2} \qquad j \geq 1 \\ \psi_{-j} &= \phi\psi_{-j-1} - \psi_{-j-2} \qquad j \geq 1 \end{aligned}$$

The solution to these difference equations are:

$$\begin{aligned} \psi_j &= d_1\lambda^j + d_2\lambda^{-j} \\ \psi_{-j} &= e_1\lambda^j + e_2\lambda^{-j} \end{aligned}$$

where the constants d_1, d_2, e_1, e_2 have yet to be determined. A sensible economic solution requires that, if the supply shock has been constant in the past and is expected to remain constant in the future, the price must be constant too. Thus, we can eliminate the exploding parts of the above solutions, setting $d_2 = 0$ and $e_2 = 0$. Next observe that both solutions must coincide for $j = 0$ which implies that $d_1 = e_1$. Denote this value by d. d can be determined by observing that the solutions must satisfy the initial value condition: $\psi_1 = \phi\psi_0 - \psi_{-1} + \frac{1}{\alpha}$. Inserting the solutions for $\psi_1, \psi_0, \psi_{-1}$ and recognizing that $\phi = \lambda + \lambda^{-1}$ leads to:

$$d\lambda = \phi d - d\lambda + \frac{1}{\alpha} \quad \Rightarrow \quad d = \frac{\alpha^{-1}}{\lambda - \lambda^{-1}}$$

The general solution to the Cobweb model with inventory represented by the difference equation (2.40) is therefore given by

$$p_t = c_1\lambda^t + c_2\lambda^{-t} + \frac{\alpha^{-1}}{\lambda - \lambda^{-1}} \sum_{j=-\infty}^{\infty} \lambda^{|j|} u_{t-j} \tag{2.41}$$

If we impose again the requirement that the price must be finite if the supply shock has always been constant and is expected to remain constant in the future, we have to set $c_1 = 0$ and $c_2 = 0$ to get the solution:

$$p_t = \frac{\alpha^{-1}}{\lambda - \lambda^{-1}} \sum_{j=-\infty}^{\infty} \lambda^{|j|} u_{t-j} \tag{2.42}$$

This implies that $\{p_t\}$ is a bounded sequence, i.e. that $(p_t) \in \ell_\infty$. In this case the price p_t is just a function of all past shocks and all expected future shocks.

Another way to represent this solution is to express p_t as

$$p_t = \lambda p_{t-1} - \alpha^{-1}\lambda \sum_{j=0}^{\infty} \lambda^j u_{t+j}.$$

In this expression the double infinite sum is replaced by a single one. This is due to the fact that the past evolution of supply is now summarized by p_{t-1} which is supposed to be known in period t. The effect of discounted expected future supply is just as before.

In order to gain a better understanding of the dynamics, we will analyze the following numerical example. In this example $\alpha = \frac{20}{9}$ and the parameters β and γ are such that $\phi = 2.05$. This implies that $\beta + \gamma = \frac{1}{9}$. The roots are then given by $\lambda = 0.8$ and $\lambda^{-1} = 1.25$. The bounded solution is then given by

$$p_t = -\sum_{j=-\infty}^{\infty} 0.8^{|j|} u_{t-j}$$

Suppose that the supply shock has been constant forever and is expected to remain constant at u. The above formula then implies that the logged price level p_t equals $-9u$ and that $I_t = 0$. Suppose that an unexpected and transitory positive supply shock of value 1 hits the market in period 0. Then according to the first panel in Figure 2.8 the price immediately falls by 1. At the same time inventories rise because prices are expected to move up in the future due to the transitory nature of the shock. Here we have a typical price movement: the price falls, but is expected to increase. After the shock the market adjusts gradually as prices rise to their old level and by running down inventories.

Consider now a different gedankenexperiment. Suppose that the shock is not unexpected, but expected to hit the market only in period 5. In this case, we see a more interesting evolution of prices and inventories. In period zero when the positive supply shock for period 5 is announced, market

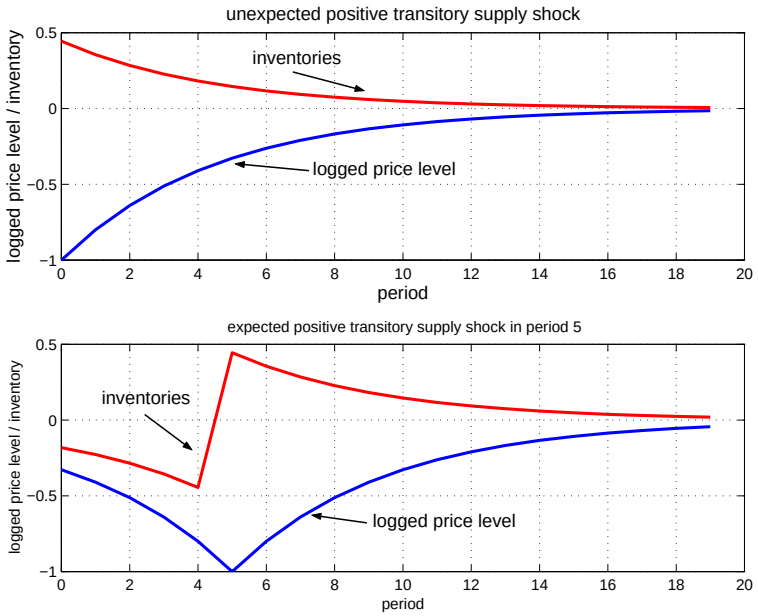


Figure 2.8: Impulse Response after a positive Supply Shock

participants expect the price to fall in the future. They therefore want to get rid of their inventories by trying to selling them already now.[12] As a result, the price and the inventories start to fall already before the supply shock actually takes place. In period 5 when the supply shock finally hits the market, market participants expect the price to move up again in the future which leads to a buildup of inventories. Note that this buildup is done when the price is low. From period 5 on, the market adjusts like in the previous case because the supply shock is again assumed to be transitory in nature.

Taylor model

In this example we analyze a simple deterministic version of Taylor's staggered wage contract model which also has a backward and forward component (see Taylor (1980) and Ashenfelter and Card (1982)).[13] In this model, half of the wages have to be contracted in each period for two periods. Thus, in each period half of the wages are renegotiated taking the wages of the other group as given. Assuming that the two groups are of equal size, wages are set according to the following rule:

$$w_t = 0.5w_{t-1} + 0.5w_{t+1} + h(y_t + y_{t+1}), \qquad h > 0. \tag{2.43}$$

[12] In our example they actually go short as $I_0 < 0$.

[13] The model could equally well be applied to analyze staggered price setting behavior.

Thus, wage setting in period t takes into account the wages of contracts still in force, w_{t-1}, and the expected wage contract in the next period, w_{t+1}. As the two groups are of equal size and power, we weight them equally by 0.5. In addition wages depend on the state of the economy over the length of the contract, here represented by aggregate demand averaged over the current and next period. The aggregate wage in period t, W_t, is then simply the average over all existing individual contract wages in place in period t:

$$W_t = \frac{1}{2}(w_t + w_{t-1}) \tag{2.44}$$

The model is closed by adding a quantity theoretic aggregate demand equation relating W_t and y_t:

$$y_t = \gamma W_t + v_t, \qquad \gamma < 0. \tag{2.45}$$

The negative sign of γ reflects the fact that in the absence of full accommodation by the monetary authority, higher average nominal wages reduce aggregate demand. v_t represents a shock to aggregate demand.

Putting equations (2.43), (2.45), and (2.44) together one arrives at a linear difference equation of order 2:

$$(1 + h\gamma)w_{t+1} - 2(1 - h\gamma)w_t + (1 + h\gamma)w_{t-1} = -2h(v_t + v_{t+1})$$

or equivalently

$$w_{t+1} - \phi w_t + w_{t-1} = Z_t \tag{2.46}$$

with $\phi = 2\frac{(1-h\gamma)}{(1+h\gamma)}$ and $Z_t = -\frac{2h}{(1+h\gamma)}(v_t + v_{t+1})$. The characteristic equation for this difference equation is

$$1 - \phi z + z^2 = 0.$$

The symmetric nature of the polynomial coefficients implies that the roots appear in pairs such that one root is the inverse of the other.[14] This means that one root, say λ_1, is smaller than one whereas the other one is greater than one, i.e. $\lambda_2 = 1/\lambda_1$. To see this note first that the discriminant is equal to $\triangle = -h\gamma > 0$. Thus, the roots are real and second that $\lambda_1\lambda_2 = 1$. If we denote λ_1 by λ then $\lambda_2 = 1/\lambda$ and we have $\phi = \lambda + \lambda^{-1}$.

Applying the superposition principle, the solution becomes

$$w_t = c_1\lambda^t + c_2\lambda^{-t} + w_t^{(p)} \tag{2.47}$$

[14]This conclusion extends to contracts longer than two periods (see Ashenfelter and Card, 1982).

where the coefficients c_1 and c_2 and a particular solution $w_t^{(p)}$ have yet to be determined. In order to eliminate explosive solutions, we set $c_2 = 0$. The other constant can then be determined by noting that (w_t) is a predetermined variable such that the wage negotiations in period one take wages from the other group negotiated in period zero as given. Thus, $c_1 = w_0 - w_0^{(p)}$. To find the particular solution, set

$$w_t^{(p)} = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

and insert this solution into the difference equation (2.47) and perform a comparison of coefficients as in the previous exercise. This leads again to two homogeneous difference equations for the coefficients (ψ_j) and (ψ_{-j}), $j \geq 1$ with solutions

$$\psi_j = d_1\lambda^j + d_2\lambda^{-j}$$
$$\psi_{-j} = e_1\lambda^j + e_2\lambda^{-j}$$

where the coefficients d_1, d_2, e_1 and e_2 have still to be determined. The elimination of explosive coefficient sequences leads to $d_2 = e_2 = 0$. Furthermore, both solutions must give the same ψ_0 so that $d_1 = e_1$. Denote this value by d, then comparing the coefficients for Z_t and noting that $\phi = \lambda + \lambda^{-1}$ leads to:

$$\psi_1 = \phi\psi_0 - \psi_{-1} + 1 \quad \Longleftrightarrow \quad d\lambda = \phi d - d\lambda + 1.$$

Therefore

$$d = \frac{1}{\lambda - \lambda^{-1}} < 0.$$

The effect of a shock to aggregate demand in period $j \geq 0$ is then

$$\frac{\partial w_{t+j}}{\partial v_t} = \psi_{-j} + \psi_{-j-1} = -\frac{2h}{1+h\gamma} d(1+\lambda)\lambda^j, \qquad j = 0, 1, 2, \ldots$$

Chapter 3

Systems of Linear Difference Equations with Constant Coefficients

3.1 Introduction

This chapter generalizes the univariate linear difference equations to systems of linear difference equations. For each variable $x_{1t}, \cdots, x_{dt}$, $d \geq 1$, we are given a nonhomogeneous difference equation of order p where each variable can, in principle, depend on all other variables with a lag. Writing each difference equation separately, the system is given by

$$\begin{aligned}
x_{1,t+1} &= a_{11}^{(1)} x_{1,t} + a_{12}^{(1)} x_{2,t} + \cdots + a_{1d}^{(1)} x_{d,t} \\
&+ \cdots + a_{11}^{(p)} x_{1,t-p+1} + a_{12}^{(p)} x_{2,t-p+1} + \cdots + a_{1d}^{(p)} x_{d,t-p+1} + b_{1t} \\
x_{2,t+1} &= a_{21}^{(1)} x_{1,t} + a_{22}^{(1)} x_{2,t} + \cdots + a_{2d}^{(1)} x_{d,t} \\
&+ \cdots + a_{21}^{(p)} x_{1,t-p+1} + a_{22}^{(p)} x_{2,t-p+1} + \cdots + a_{2d}^{(p)} x_{d,t-p+1} + b_{2t} \\
&\cdots \\
x_{d,t+1} &= a_{d1}^{(1)} x_{1,t} + a_{d2}^{(1)} x_{2,t} + \cdots + a_{dd}^{(1)} x_{d,t} \\
&+ \cdots + a_{d1}^{(p)} x_{1,t-p+1} + a_{d2}^{(p)} x_{2,t-p+1} + \cdots + a_{dd}^{(p)} x_{d,t-p+1} + b_{dt}
\end{aligned}$$

Using matrix notation this equation system can be written more compactly as

$$x_{t+1} = A_1 x_t + A_2 x_{t-1} + \cdots + A_p x_{t-p+1} + b_t, \qquad A_p \neq 0, \tag{3.1}$$

where b_t denotes an d-vector of exogenous variables and where A_i, $i =$

$1, 2, \ldots, p$, denote the $d \times d$-matrices $A_k = \left(a_{ij}^{(k)}\right)_{i,j=1,2,\ldots,d}$ for $k = 1, 2, \ldots, p$. The solution of this difference equation is based again on the same principles as in the univariate case (see page 25). Before doing so we show how to reduce this p-th order system to a first order system.

Any system of order p can be rewritten as a system of order one by enlarging the state space from $\mathbb{R}^d$ to $\mathbb{R}^{dp}$. To see this, define a new variable y_t as the stacked vectors $x_t, x_{t-1}, \cdots, x_{t-p+1}$. This new variable then satisfies the following first order system:

$$\begin{aligned} y_{t+1} &= \begin{pmatrix} x_{t+1} \\ x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \\ x_{t-p+2} \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & A_3 & \ldots & A_{p-1} & A_p \\ I_n & 0 & 0 & \ldots & 0 & 0 \\ 0 & I_n & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & I_d & 0 \end{pmatrix} \begin{pmatrix} x_t \\ x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p+2} \\ x_{t-p+1} \end{pmatrix} + \begin{pmatrix} b_t \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \\ &= Ay_t + \tilde{b}_t \end{aligned} \tag{3.2}$$

where $\tilde{b}_t$ is redefined to be $\begin{pmatrix} b_t' & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}'$. I_d denotes the identity matrix of dimension d. The matrix A is a $dp \times dp$ matrix called the *companion matrix* of (3.1).[1] Thus, multiplying out the equation system (3.2) one can see that the first equation gives again the original equation (3.1) whereas the remaining $p - 1$ equations are just identities. The study of a p-th order system can therefore always be reduced to a first order system.

Properties of the Companion Matrix in the Univariate Case

The one-dimensional difference equation of order p, equation (2.1), can also be written in this way as a first order system of dimension p. The companion matrix is given in this case by

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{p-1} & a_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{3.3}$$

For this companion matrix, we can derive the following properties:

- The companion matrix is nonsingular if and only if $a_p \neq 0$.

[1] The literature distinguishes four forms of companion matrices depending on whether the A_i's appear in the first, as in equation (3.2), or last row or first or last column.

- The characteristic polynomial of the companion matrix is: $\mathcal{P}(\lambda) = \lambda^p - a_1\lambda^{p-1} - \cdots - a_{p-1}\lambda - a_p$. Thus, the roots of the characteristic polynomial of the companion matrix are just the inverses of the roots of the characteristic polynomial of the difference equation (2.31).
- The geometric multiplicity of each eigenvalue equals 1, i.e. there is only one independent eigenvector for each λ_i. These eigenvectors are of the form $\left(\lambda_i^{p-1}, \lambda_i^{p-2}, \cdots, \lambda_i, 1\right)'$. Thus, in this situation there is no need to rely on the Jordan canonical form (see Section 3.2.2).

3.2 First Order Systems

The introduction above demonstrated that the first order system of difference equations encompasses single as well as systems of difference equations of order p. We therefore reduce our analysis to the first order system of difference equations:

$$x_{t+1} = Ax_t + b_t, \qquad A \neq 0, \quad t \in \mathbb{Z}, \tag{3.4}$$

where $x_t \in \mathbb{R}^d$ and A is a $d \times d$ matrix. The nonautonomous part is represented by the d-vector b_t which corresponds to a vector of exogenous variables. In general, solutions may not exist for negative times. In fact, when A has not full rank, then for points in the range of A there exists x_{-1} such that $x_0 = Ax_{-1}$ with x_{-1} being not unique.[2] Thus, for simplicity, we restrict ourself to the case where A is nonsingular. Hence the maintained assumption throughout this monograph is that $A \in \mathbb{GL}(d)$, the set of invertible real $d \times d$ matrices.

As in the univariate case, the superposition principle also holds in the multivariate case. Thus, any solution is composed as a sum of the general solution to the homogeneous system and a particular solution to the nonhomogeneous system. We therefore establish first an explicit solution formula for the homogeneous system and then show how get a particular solution for the nonhomogeneous system.

3.2.1 Homogeneous First Order Systems

As in the one-dimensional case, we start the analysis with the discussion of the homogeneous equation:

$$x_{t+1} = Ax_t, \qquad A \in \mathbb{GL}(d) \text{ and } t \in \mathbb{Z}. \tag{3.5}$$

[2] A singular A matrix may be interpreted as a system which encompasses some redundant variables. See Section 3.5.

We immediately see that starting with some initial vector $x_0 = x$, all subsequent values of x_t, $t > 0$, are uniquely determined. The invertibility of A implies that also all past values are uniquely determined. Indeed, $x_{-(t+1)} = A^{-1}x_{-t}$. Hence, for any initial condition $x_0 = x$, we have the solution

$$x_t = A^t x_0 = A^t x, \qquad t \in \mathbb{Z}.$$

To highlight the dependency on the initial condition, we write $x_t = \varphi(t, x) = A^t x$. Note that the invertibility of A implies that solutions are unique up to x. Indeed, $x \neq y$ implies $x_t = \varphi(t, x) = A^t x \neq y_t = \varphi(t, y) = A^t y$.

Linear Dependence/Independence Suppose that we have two solutions to the homogeneous system (3.5), $x_t^{(1)}$ and $x_t^{(2)}$. Then, it is clear that any linear combination of these two solutions, $c_1 x_t^{(1)} + c_2 x_t^{(2)}$, is also a solution. Thus, the set of all solutions to the homogeneous system (3.5) forms a linear space. As in the univariate case, we analyze the algebraic structure of this space and determine its dimension.

The definition for the linear independence of r solutions is:

Definition 3.1 (Linear Dependence, Linear Independence)**.** The sequences $(x^{(1)}), (x^{(2)}), \cdots, (x^{(r)})$ with $r \geq 1$ are said to be *linearly dependent* if there exist constants $c_1, c_2, \ldots, c_r \in \mathbb{R}$, not all zero, such that

$$c_1 x_t^{(1)} + c_2 x_t^{(2)} + \cdots + c_r x_t^{(r)} = 0 \qquad \text{for all } t \in \mathbb{Z}.$$

This definition is equivalent to saying that there exists a nontrivial linear combination of the solutions which is zero. If the solutions are not linearly dependent, they are said to be *linearly independent*.

For given r sequences $(x^{(1)}), (x^{(2)}), \cdots, (x^{(r)})$, we can define their *Casoratian* matrix $\mathcal{C}(t)$:

$$\mathcal{C}(t) = \begin{pmatrix} x_{1t}^{(1)} & x_{1t}^{(2)} & \cdots & x_{1t}^{(r)} \\ x_{2t}^{(1)} & x_{2t}^{(2)} & \cdots & x_{2t}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{rt}^{(1)} & x_{rt}^{(2)} & \cdots & x_{rt}^{(r)} \end{pmatrix}$$

The Casoratian matrix is closely related to the issue whether or not the sequences are independent.

Lemma 3.1. If $\det \mathcal{C}(t)$ of r sequences $(x^{(i)})$, $1 \leq i \leq r$, is different from zero for at least one $t_0 \in \mathbb{Z}$, then $(x^{(i)})$, $1 \leq i \leq r$, are linearly independent.

Proof. Suppose that $(x^{(i)})$, $1 \leq i \leq r$, are linearly dependent. Then by definition there exists a nonzero vector c such that $\mathcal{C}(t)c = 0$ for all t. In particular, $\mathcal{C}(t_0)c = 0$. This stands, however, in contradiction with the assumption $\det \mathcal{C}(t_0) \neq 0$. □

Note that the converse is not true as can be seen from the following example: $x_t^{(1)} = \begin{pmatrix} 1 \\ t \end{pmatrix}$ and $x_t^{(2)} = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, $t \in \mathbb{N}$. These two sequences are linearly independent, but $\det \mathcal{C}(t) = 0$ for all $t \geq 0$. The converse of Lemma 3.1 is true if the sequences are solutions to the homogeneous equation (3.5).

Lemma 3.2. If $(x^{(i)})$, $1 \leq i \leq r$, are r linearly independent solutions of the homogeneous system (3.5), then $\det \mathcal{C}(t) \neq 0$ for all $t \in \mathbb{Z}$.

Proof. Suppose there exists a t_0 such $\det \mathcal{C}(t_0) = 0$. This implies that there exists a nonzero vector c such that $\mathcal{C}(t_0)c = \sum_{i=1}^{r} c_i x_{t_0}^{(i)} = 0$. Because the $x_t^{(i)}$ are solutions of the system (3.5) so is the linear combination $y_t = \sum_{i=1}^{r} c_i x_t^{(i)}$. For this solution $y_{t_0} = 0$ thus $y_t = 0$ for all t because the invertibility of A (uniqueness of the solutions). As the solutions are, however, linearly independent c must be equal to 0 which stands in contradiction to $c \neq 0$. □

We can combine the two Lemmas to obtain the following theorem.

Theorem 3.1. The solutions $(x^{(i)})$, $1 \leq i \leq r$, of the homogeneous system (3.5) are linearly independent if and only if there exists $t_0 \geq 0$ such that $\det \mathcal{C}(t_0) \neq 0$.

Fundamental Matrix The above Theorem implies that the d solutions $\Phi_t^{(i)} = \varphi(t, e^{(i)})$, $1 \leq i \leq d$, of the homogeneous system (3.5) where

$$\Phi_0^{(i)} = \varphi(0, e^{(i)}) = e^{(i)} = \begin{pmatrix} 0 & \cdots & 0 & \underbrace{1}_{\text{i-th element}} & 0 & \cdots & 0 \end{pmatrix}', \qquad 1 \leq i \leq d, \tag{3.6}$$

are linearly independent. Thus, we have at least d linearly independent solutions. Now suppose that we are given any solution $x_t = \varphi(t, x)$ of the homogeneous system (3.5). Then it is easy to see that we can express x_t as $x_t = \sum_{i=1}^{d} x_{i,0} \Phi_t^{(i)}$ where $\Phi_t^{(i)} = \varphi(t, e^{(i)})$. As the solutions are uniquely determined, we have thus shown that the space of all solutions to the homogeneous system (3.5) is a linear space of dimension d. Thus, any solution can be written as

$$x_t = \varphi(t, x) = \sum_{i=1}^{d} x_{i,0} \Phi_t^{(i)} = \Phi(t)x \tag{3.7}$$

where $\Phi(0) = I_d$ and $\Phi(t) = (\Phi_t^{(1)}, \ldots, \Phi_t^{(d)})$.

Because each column of the matrix function $\Phi(t)$ is a solution to the homogeneous system (3.5), $\Phi(t)$ satisfies the *homogeneous linear matrix system*:

$$\Phi(t+1) = A\Phi(t) \tag{3.8}$$

This leads to the following definitions.

Definition 3.2 (Fundamental Matrix)**.** Any sequence of nonsingular $d \times d$ matrices $\Phi(t)$ which satisfies the homogeneous matrix system (3.8) is called a *fundamental matrix*. If in addition the matrix satisfies $\Phi(0) = I_d$ then it is called a *principal fundamental matrix*.

Note that if $\mathcal{V}(t)$ is any fundamental matrix then $\Phi(t) = \mathcal{V}(t)\mathcal{V}^{-1}(0)$ is a principal fundamental matrix. Note also if $\mathcal{V}(t)$ is a fundamental matrix then $\mathcal{V}(t)C$ is also a fundamental matrix where C is any nonsingular matrix. This implies that there are infinitely many fundamental matrices for a given homogeneous matrix system. There is, however, only one principal fundamental matrix because the matrix difference equation (3.8) uniquely determines all subsequent matrices once an initial matrix is given. In the case of a principal fundamental matrix this initial matrix is the identity matrix.

On a more general note define $\Phi(t, s)$ as

$$\Phi(t, s) = \mathcal{V}(t)\mathcal{V}^{-1}(s), \qquad t, s \in \mathbb{Z}$$

where $\mathcal{V}(t)$ is any fundamental matrix of the homogeneous matrix system (3.8). It is easy to verify that, for all $t, s \in \mathbb{Z}$, $\Phi(t, s)$ has the following properties:

(i) $\Phi(t+1, s) = A\Phi(t, s)$. Hence, $\Phi(t, s)$ is a solution of the matrix difference equation, therefore a fundamental matrix;

(ii) $\Phi(t, 0)$ is a principal fundamental matrix;

(iii) $\Phi^{-1}(t, s) = \Phi(s, t)$;

(iv) $\Phi(t, s) = A^{t-s}$ when the system is given by (3.5).

In the case of the linear homogeneous system (3.5) $\Phi(t) = A^t$ is the principal fundamental matrix. Thus, any solution to the homogeneous system (3.5) has the form:

$$x_t = \Phi(t)x = A^t x \tag{3.9}$$

where $x \in \mathbb{R}^d$ is a given vector.

Based on $\Phi(t, s)$, we define *Green's matrix* as the kernel

$$\Gamma(t, s) = \Phi(t, r)\Phi^{-1}(s, r), \qquad r, s, t \in \mathbb{Z}.$$

Lemma 3.3. For all $r, s, t \in \mathbb{Z}$ Green's matrix satisfies:

(i) $\Gamma(t,t) = I_d$;

(ii) $\Gamma^{-1}(t,s) = \Gamma(s,t)$;

(iii) $\Gamma(t+1,s) = A\Gamma(t,s)$ and $\Gamma(t,s+1) = \Gamma(t,s)A^{-1}$;

(iv) $\Gamma(t,s) = \Gamma(t,r)\Gamma(r,s)$;

(v) $\Gamma(t,s) = A^{t-s}$ when the system is given by (3.5).

3.2.2 Solution Formula for Homogeneous Systems

In order to find the explicit solution formulas of the homogeneous system (3.5) and to understand its properties, we need to find an expression for A^t. In analogy to the univariate case, assume the solution is again a power of some number $\lambda \neq 0$ times a constant vector $x \neq 0$, i.e. $x_t = \lambda^t x$. Inserting this guess into the defining homogeneous equation leads to

$$x_{t+1} = \lambda^{t+1}x = A\lambda^t x.$$

Dividing by λ^t this implies

$$\lambda x = Ax \quad \text{or} \quad (A - \lambda I_d)x = 0.$$

Thus, for $x_t = \lambda^t x$ to be a solution, λ and x must be an eigenvalue, respectively an eigenvector of A. If there are d distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_d\}$, Theorem 2.3 implies that $\{\lambda_1^t, \lambda_2^t, \ldots, \lambda_d^t\}$ are linearly independent. Hence the solutions form a d-dimensional linear space. The understanding of the homogeneous system therefore requires an analysis of the eigenvalues and eigenvectors of the matrix A.[3] It is useful to distinguish in this context two cases.

Distinct Eigenvalues

If all the eigenvalues, $\lambda_1, \cdots, \lambda_d$ of A, are distinct, then A is diagonalizable, i.e. *similar* to a diagonal matrix. Thus, there exists a nonsingular matrix Q such that $Q^{-1}AQ = \Lambda$ where $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_d)$. The columns of Q

[3]Appendix C.3 provides an outline of the basic properties of eigenvalues and eigenvectors. For details consult standard textbooks on linear algebra, like Meyer (2000) and Strang (2003).

consist of the d linearly independent eigenvectors of A. With this similarity transformation in mind A^t can be immediately computed as:

$$\begin{aligned} A^t &= \underbrace{Q\Lambda Q^{-1}Q\Lambda Q^{-1}\cdots Q\Lambda Q^{-1}}_{\text{t times}} = Q\Lambda^t Q^{-1} \\ &= Q\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{pmatrix}^t Q^{-1} = Q\begin{pmatrix} \lambda_1^t & 0 & \cdots & 0 \\ 0 & \lambda_2^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d^t \end{pmatrix} Q^{-1} \end{aligned}$$

In the case of distinct eigenvalues, it is easy to proof the following theorem.

Theorem 3.2. If the spectrum of A, $\sigma(A) = \{\lambda_1, \cdots, \lambda_d\}$, consists of d distinct eigenvalues with corresponding eigenvectors q_i, $i = 1, \ldots, d$, then the set

$$x_t^{(i)} = \lambda_i^t q_i, \qquad i = 1, \ldots, d,$$

represents a fundamental set of solutions to the homogeneous system (3.5).

Proof. As q_i is an eigenvector of A corresponding to λ_i, we have

$$x_{t+1}^{(i)} = \lambda_i^{t+1} q_i = \lambda_i \lambda_i^t q_i = A\lambda_i^t q_i = Ax_t^{(i)}, \qquad t \geq 0.$$

The third equality follows from $AQ = Q\Lambda$. Thus, the $x_t^{(i)}$, $i = 1, \cdots, d$, are solutions to the homogeneous system (3.5). In addition, we have that the determinant of the corresponding Casoratian matrix evaluated at $t = 0$ is $\det \mathcal{C}(0) = \det(q_1, \ldots, q_n) = \det Q \neq 0$ because Q consists of d linearly independent eigenvectors and is therefore nonsingular. Thus, according to Theorem 3.1 these solutions are linearly independent. □

The general solution to the homogeneous system (3.5) can be written as

$$x_t = \sum_{i=1}^{d} c_i \lambda_i^t q_i, \qquad t \in \mathbb{Z}, \tag{3.10}$$

for some constants $c_1, \cdots, c_d$. If an eigenvalues is complex, the entries of the corresponding eigenvector must be complex too in order to ensure real values for x_t. From the above formula it is clear that the asymptotic growth rate of x_t is governed by the eigenvalue with the largest modulus, provided that the corresponding constant is nonzero. See Section 3.2.3 for a general statement.

A geometric interpretation of the general solution (3.10) is obtained by decomposing A into the sum of projector matrices according to the Spectral Theorem C.1 given in Appendix C.3.

If A is diagonalizable, the Spectral Theorem implies that we can decompose any $x \in \mathbb{R}^d$ into s components $P_1x, \ldots, P_sx$ such that, for $j = 1, \ldots, s$, P_jx_0 is projected onto the subspace $\mathbf{N}(A - \lambda_j I_d)$. Corollary C.1 then implies that this component grows at the exponential rate $\log|\lambda_j| = \lim_{\pm t\to\infty} \frac{1}{t} \log\|\lambda_j^t P_j x_0\|$. Thus, this component converges to zero if if $|\lambda_j| < 1$ and diverges if $|\lambda_j| > 1$. A systematic exposition of this aspect is given in Section 3.2.3 below and will become particularly relevant in the analysis of time–varying systems treated in Chapter 6.

Another way to understand the result of Theorem 3.2 is to observe that the similarity transformation actually uncouples the interrelated system in x_t into d unrelated univariate first order difference equations. This decoupling is achieved by a change of basis of the state space which corresponds to the variable transformation $y_t = Q^{-1}x_t$:

$$\begin{aligned} y_{t+1} &= Q^{-1}x_{t+1} = Q^{-1}AQQ^{-1}x_t = \Lambda y_t \\ \begin{pmatrix} y_{1,t+1} \\ \vdots \\ y_{d,t+1} \end{pmatrix} &= \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_d \end{pmatrix} \begin{pmatrix} y_{1,t} \\ \vdots \\ y_{d,t} \end{pmatrix} \end{aligned}$$

Thus, through this transformation we have obtained d unrelated univariate first order difference equations in $y_{i,t}$, $i = 1, \ldots, d$:

$$\begin{aligned} y_{1,t+1} &= \lambda_1 y_{1,t} \\ &\cdots \\ y_{d,t+1} &= \lambda_d y_{d,t} \end{aligned}$$

These equations can be solved one-by-one by the methods discussed in chapter 2. The general solutions to these univariate first order homogeneous equations are therefore $y_{i,t} = c_i\lambda_i^t$, $i = 1, \cdots, d$. Transforming the system in y_t back to the original system by multiplying y_t from the left with Q yields exactly the solution in equation (3.10).

This change of basis, thus, transforms a complicated system into a simpler one such that there is an exact correspondence between the original system and the transformed one. We can visualize this insight by viewing A and Λ as maps from $\mathbb{R}^d$ to $\mathbb{R}^d$. This leads to the commutative diagram below.

$$\begin{array}{ccc} \mathbb{R}^d & \xrightarrow{A} & \mathbb{R}^d \\ {\scriptstyle Q^{-1}}\downarrow & & \downarrow{\scriptstyle Q^{-1}} \\ \mathbb{R}^d & \xrightarrow{\Lambda} & \mathbb{R}^d \end{array}$$

The commutativity is expressed by the identity $\Lambda Q^{-1} = Q^{-1}A$. On a more general note, we say that A is *conjugate* to Λ via the map Q^{-1}. A more

sophisticated application of conjugation is given by linearizing nonlinear systems. See the Hartman–Grobman Theorem 1.5 presented in Chapter 1.3.2.

Repeated Eigenvalues

The situation with repeated eigenvalues is more complicated. Before dealing with the general case, note that even with repeated eigenvalues the matrix A can be diagonalizable. This is, for example, the case for *normal matrices*, i.e. matrices for which $AA' = A'A$. Examples of normal matrices include symmetric matrices ($A = A'$), skew symmetric matrices ($A = -A'$) and unitary or orthogonal matrices ($AA' = A'A = I$). More generally, A is diagonalizable if and only if, for each eigenvalue, the algebraic multiplicity (the multiplicity as a root of the characteristic polynomial) equals the geometric multiplicity (the maximum number of linearly independent eigenvectors). It this case the eigenvalues are called *semisimple*. An eigenvalue is called *simple* if its algebraic multiplicity is one.[4] If the maximum number of linearly independent eigenvectors corresponding to some eigenvalue is strictly less than its algebraic multiplicity, the matrix is called *defective*.

As A is not, in general, diagonalizable, we have to use its Jordan canonical form to find an expression for the solution of equation (3.5).[5] For every matrix A with distinct eigenvalues $\sigma(A) = \{\lambda_1, \cdots, \lambda_s\}$, there exists a nonsingular matrix Q such that A can be reduced to a block diagonal matrix J by a similarity transformation, i.e. $Q^{-1}AQ = J$:

$$J = Q^{-1}AQ = \begin{pmatrix} J(\lambda_1) & 0 & \cdots & 0 \\ 0 & J(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J(\lambda_s) \end{pmatrix}$$

The *Jordan segments* $J(\lambda_i)$, $i = 1, \cdots, s$, consist of t_i *Jordan blocks*, $J_j(\lambda_i)$, $j = 1, \cdots, t_i$, where t_i is the dimension of the nullspace of $A - \lambda_i I$, i.e. $t_i = \dim \mathbf{N}(A - \lambda_i I)$. Thus, t_i is the number of independent eigenvectors corresponding to the eigenvalue λ_i. Each Jordan segment $J(\lambda_i)$ has a block

[4] Obviously, a simple eigenvalue is always semisimple.

[5] A detailed treatment of the Jordan canonical form can be found, for example, in Meyer (2000). The current exposition uses the complex Jordan form. There is, however, an equivalent presentation based on the real Jordan form (see Colonius and Kliemann, 2014).

diagonal structure:

$$J(\lambda_i) = \begin{pmatrix} J_1(\lambda_i) & 0 & \cdots & 0 \\ 0 & J_2(\lambda_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{t_i}(\lambda_i) \end{pmatrix}.$$

The Jordan blocks themselves are of the following form:

$$J_j(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix} = \lambda_i I + N \tag{3.11}$$

where

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The square matrix N is a *nilpotent matrix*, i.e. $N^k = 0$ if k is the dimension of N. The dimension of the largest Jordan block in the Jordan segment $J(\lambda_i)$ is called the *index of the eigenvalue* λ_i, denoted by $k_i = \text{index}(\lambda_i)$. If $k_i = 1$, the matrices $J_j(\lambda_i)$ are just scalars equal to λ_i. Thus, $J(\lambda_i)$ is a diagonal matrix of dimension t_i with λ_i on the diagonal. Therefore, if the index of every eigenvalue equals one, the matrix is diagonalizable.

Given these preliminaries it is now a straightforward task to compute $A^t = QJ^tQ^{-1} = Q \text{ diag}\left(J^t(\lambda_1), \cdots, J^t(\lambda_s)\right) Q^{-1}$ where the t-th power of a Jordan segment $J^t(\lambda_i)$ is just $J^t(\lambda_i) = \text{diag}\left(J_1^t(\lambda_i), \cdots, J_{t_i}^t(\lambda_i)\right)$. The t-th power of a Jordan block $J_j(\lambda_i)$, $j = 1, \cdots, t_i$, is given by the expression:

$$\begin{aligned} J_j(\lambda_i)^t &= (\lambda_i I + N)^t \\ &= \lambda_i^t I + \binom{t}{1}\lambda_i^{t-1}N + \binom{t}{2}\lambda_i^{t-2}N^2 + \cdots + \binom{t}{k-1}\lambda_i^{t-k+1}N^{k-1} \\ &= \begin{pmatrix} \lambda_i^t & \binom{t}{1}\lambda_i^{t-1} & \binom{t}{2}\lambda_i^{t-2} & \cdots & \binom{t}{k-1}\lambda_i^{t-k+1} \\ 0 & \lambda_i^t & \binom{t}{1}\lambda_i^{t-1} & \cdots & \binom{t}{k-2}\lambda_i^{t-k+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ & & & & \binom{t}{1}\lambda_i^{t-1} \\ 0 & 0 & 0 & \cdots & \lambda_i^t \end{pmatrix} \end{aligned} \tag{3.12}$$

where N is the nilpotent matrix of size corresponding to the Jordan block.

Real Jordan Decomposition

When the eigenvalues are complex some entries of the corresponding eigenvectors must be complex too to achieve real values for x_t. In order to avoid this intricacy, one may compute the *real Jordan decomposition* instead. This decomposition relies on the fact that the complex eigenvalues appear as a conjugate pairs. One may then construct Jordan blocks of the form

$$J_j\left(\lambda_i = \alpha_i \pm \beta_i\right) = \begin{pmatrix} D_i & I_2 & 0 & \cdots & 0 & 0 \\ 0 & D_i & I_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_i & I_2 \\ 0 & 0 & 0 & \cdots & 0 & D_i \end{pmatrix}$$

where $D_i = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}$ and I_2 is the identity matrix of order two. With this modification one obtains a factorization $J^{\mathbb{R}} = Q^{-1}AQ$ where only real entries appear in $J^{\mathbb{R}}$ and Q.

3.2.3 Lyapunov spaces

The asymptotic behavior of solutions of linear homogeneous difference equations provides a key to understand the connection between linear algebra and difference equations (dynamical systems). An aspect which will become key when analyzing systems with time-varying coefficients (see Chapter 6). Here we follow closely the exposition of Colonius and Kliemann (2014) and define the Lyapunov exponent.

Definition 3.3 (Lyapunov exponent)**.** Denote by $x_t = \varphi(t, x) = A^t x$, $t \in \mathbb{Z}$, a solution of the linear homogeneous difference equation (3.5). Its *Lyapunov exponent* or *exponential growth rate* is defined as

$$\lambda(x) = \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi(t, x)\|, \qquad x \neq 0,$$

where log denotes the natural logarithm and $\|.\|$ the Euclidean norm in $\mathbb{R}^d$.

From the definition it is obvious that vectors with different Lyapunov exponents are independent. Thus, there are at most d distinct Lyapunov exponents.[6] Before analyzing the general theory, It is instructive to consider

[6] Additional properties of Lyapunov exponents can be found in Arnold (2003, section 3.2.1)

the univariate case $x_{t+1} = ax_t$, $0 \neq a \in \mathbb{R}$, the solutions are $\varphi(t, x) = a^t x$. The Lyapunov exponent is therefore given as a limit:

$$\lambda(x) = \lim_{t\to\pm\infty} \frac{1}{t} \log(|a^t x|) = \lim_{t\to\pm\infty} \frac{1}{t} \log(|a|^t) + \lim_{t\to\pm\infty} \frac{1}{t} \log(|x|) = \log(|a|).$$

Thus, we have convergence to zero if and only if $\lambda(x) < 0$ or equivalently $|a| < 1$.

The (real) Jordan decomposition induces a splitting of the state space $\mathbb{R}^n$ into a direct sum of ℓ subspaces so–called *Lyapunov spaces*:

$$\mathbb{R}^n = L(\lambda_1) \oplus \ldots \oplus L(\lambda_\ell).$$

Thereby each Lyapunov space $L(\lambda_j)$, $j = 1, \ldots, \ell$, is constructed as follows. Suppose that there are $1 \leq \ell \leq d$ distinct moduli of the eigenvalues of A. Denote the logarithm these moduli by λ_j and order them as $\lambda_1 > \ldots > \lambda_\ell$.[7] Then define $L_j = L(\lambda_j)$ as the direct sum of the real generalized eigenspaces associated to eigenvalues with logged modulus equal to λ_j. It can then be shown (see Colonius and Kliemann, 2014, theorem 1.5.6) that the Lyapunov exponents $\lambda(x)$ are given by the logarithms λ_j of the moduli of the eigenvalues of A. Moreover, for an orbit $\varphi(., x)$ with $x \neq 0$ one has

$$\lambda(x) = \lim_{t\to\pm\infty} \frac{1}{t} \log \|\varphi(t, x)\| = \lambda_j \quad \text{if and only if} \quad x \in L(\lambda_j).$$

For the characterization of Lyapunov spaces it is important to consider the limit to $+\infty$ and $-\infty$. Take, for example, $x = x_1 + x_2$ where $x_i \in L(\lambda_i)$, $\lambda_1 < \lambda_2$, and $x \neq 0$, then $\lambda(x) = \lambda_2$. A refined version of this result with respect to limits in positive and negative times can be obtained using flags of subspaces (see Colonius and Kliemann, 2014, theorem 1.5.8).

3.2.4 Nonhomogeneous First Order System

Consider now the first order nonhomogeneous system

$$x_{t+1} = Ax_t + b_t \tag{3.13}$$

where b_t is assumed to be bounded. As in the one-dimensional case, the superposition principle also holds in the multivariate case. Suppose that there exist two solutions to the nonhomogeneous system (3.13), $x_t^{(1)}$ and $x_t^{(2)}$.

[7] It should be clear from the context whether λ denotes an eigenvalue of a matrix or a Lyapunov exponent. As the two are concepts are intimately related we use the same notation for both.

Then one can easily verify that $x_t^{(1)} - x_t^{(2)}$ is a solution to the homogeneous system (3.5). Thus, the superposition principle implies that $x_t^{(1)} - x_t^{(2)} = A^t c$ for some vector c.

Theorem 3.3. Every solution x_t to the first order nonhomogeneous system (3.13) can be represented as the sum of the general solution to the homogeneous system (3.5), $(x_t^{(g)})$, and a particular solution to the nonhomogeneous system (3.13), $(x_t^{(p)})$:

$$x_t = x_t^{(g)} + x_t^{(p)} = A^t c + x_t^{(p)}. \tag{3.14}$$

One can try to find a particular solution by iterating the difference equation *backwards*:

$$\begin{aligned} x_t &= Ax_{t-1} + b_{t-1} \\ x_t &= A(Ax_{t-2} + b_{t-2}) + b_{t-1} = A^2 x_{t-2} + Ab_{t-2} + b_{t-1} \\ &\cdots \\ x_t &= A^t x_0 + A^{t-1} b_0 + A^{t-2} b_1 + \cdots + Ab_{t-2} + b_{t-1} \\ &= A^t x_0 + \sum_{j=0}^{t-1} A^j b_{t-1-j} \end{aligned}$$

By taking limits to infinity, this suggests

$$x_t^{(p)} = \sum_{j=0}^{\infty} A^j b_{t-1-j}$$

as a candidate for a particular solution. So that the general solution to the nonhomogeneous equation would become

$$x_t = A^t c + \sum_{j=0}^{\infty} A^j b_{t-1-j}. \tag{3.15}$$

This idea will, however, only work out if the modulus of all eigenvalues of A are strictly smaller than one. In this case, the infinite sum converges and the particular solution is well–defined. The initial value problem can then be solved by setting $c = x_0 - \sum_{j=0}^{\infty} A^j b_{-1-j}$. If b_t is constant and equal to b, the constant c equals the deviation from the steady state in period 0, i.e. $x_0 - x^*$ where $x^* = (I_d - A)^{-1} b$ is the steady state of the system.

This procedure will, however, fail if some or all eigenvalues are outside the unit circle. Suppose that all eigenvalues are strictly greater than one in

absolute value. In this case we can take up the ideas from the univariate case (see f.e. the equity price model and Cagan's hyperinflation model in 2.3) and iterate the difference equation *forward* k times:

$$\begin{aligned}
x_t &= A^{-1}x_{t+1} - A^{-1}b_t \\
x_t &= A^{-1}(A^{-1}x_{t+2} - A^{-1}b_{t+1}) - A^{-1}b_t \\
&= A^{-2}x_{t+2} - A^{-1}b_t - A^{-2}b_{t+1} \\
&\dots \\
x_t &= A^{-k}x_{t+k} - A^{-1}b_t - A^{-2}b_{t+1} - \dots + A^{-k}b_{t+k-1} \\
&= A^{-k}x_{t+k} - \sum_{j=1}^{k} A^{-j}b_{t-1+j}
\end{aligned}$$

Taking k to infinity suggests

$$x_t^{(p)} = -\sum_{j=1}^{\infty} A^{-j}b_{t-1+j} \tag{3.16}$$

as candidate for a particular solution. This solution is well–defined as the infinite sum converges. The unique nonexploding solution is then obtained by setting $c = 0$. Hence, $x_t = x_t^{(p)} = -\sum_{j=1}^{\infty} A^{-j}b_{t-1+j}$.

Consider next the case where A has the block diagonal form

$$A = \begin{pmatrix} A^- & 0 \\ 0 & A^+ \end{pmatrix}$$

where A^- has eigenvalues with moduli strictly smaller than one whereas A^+ has eigenvalues with moduli strictly greater than one. Partition the b_t conformably as $b_t = (b_t^{-\prime}, b_t^{+\prime})'$. A well–defined particular solution is then obtained by combining the backward and the forward procedure:

$$x_t^{(p)} = \begin{pmatrix} \sum_{j=0}^{\infty} A^j b_{t-1-j}^- \\ -\sum_{j=1}^{\infty} A^{-j} b_{t-1+j}^+ \end{pmatrix}$$

The general solution of the nonhomogeneous equation then becomes

$$x_t = \begin{pmatrix} A^- & 0 \\ 0 & A^+ \end{pmatrix}^t \begin{pmatrix} c^- \\ c^+ \end{pmatrix} + \begin{pmatrix} \sum_{j=0}^{\infty} A^j b_{t-1-j}^- \\ -\sum_{j=1}^{\infty} A^{-j} b_{t-1+j}^+ \end{pmatrix} \tag{3.17}$$

where the vector c is partitioned conformably as $c = (c^{-\prime}, c^{+\prime})'$. Thus, x_t will remain bounded if and only if $c^+ = 0$. Whether this assumption makes

sense depends on the particular economic model at hand. Finally, c^- has to be determined from some initial value condition. If c^- cannot be uniquely determined, the system becomes indeterminate.

In general, A will not be of the nice block diagonal form as above and will have some eigenvalues smaller than one in absolute value and some eigenvalues larger than one in absolute value. Thus, we will have a mixture of backward and forward solutions. Moreover, for this strategy to work and to arrive at a non–diverging solution, the number of stable roots must match the number of initial conditions. We will analyze the general case Section 3.4.

Variation of Constants Formula Another general way to derive particular solutions of the nonhomogeneous equation is by applying the so-called *variation of constants formula*. This approach makes explicit use of the fundamental matrix and Green's matrix. From equation (3.9) the solution of the homogeneous equation can be written as $x_t = \Phi(t)c$ for some constant c where $\Phi(t)$ is a fundamental matrix. Allowing c to vary over time it is possible to generate a solution of nonhomogeneous equation by setting $x_t = \Phi(t)c_t$. If this should be a solution of the nonhomogeneous,

$$x_{t+1} = \Phi(t+1)c_{t+1} = Ax_t + b_t = A\Phi(t)c_t + b_t = \Phi(t+1)c_t + b_t.$$

This implies

$$\begin{aligned} \Phi(t+1)(c_{t+1} - c_t) &= b_t \\ c_{t+1} - c_t &= \Phi^{-1}(t+1)b_t \end{aligned}$$

Because $c_t = (c_t - c_{t-1}) + \cdots + (c_{s+1} - c_s) + c_s$, for $t > s$,

$$\begin{aligned} c_t &= c_s + \sum_{j=s}^{t-1} \Phi^{-1}(j+1)b_j = c_s + \sum_{j=s+1}^{t} \Phi^{-1}(j)b_{j-1} \\ &= c_s + \sum_{j=0}^{t-1-s} \Phi^{-1}(t-j)b_{t-1-j}. \end{aligned}$$

Hence, multiplying by $\Phi(t)$

$$x_t = \Phi(t)c_t = \Phi(t)c_s + \Phi(t) \sum_{j=0}^{t-1-s} \Phi^{-1}(t-j)b_{t-1-j}.$$

Because $x_s = \Phi(s)c_s$,

$$\begin{aligned} x_t &= \Phi(t)\Phi^{-1}(s)x_s + \sum_{j=0}^{t-1-s} \Phi(t)\Phi^{-1}(r)\Phi(r)\Phi^{-1}(t-j)b_{t-1-j} \\ &= \Phi(t,s)x_s + \sum_{j=0}^{t-1-s} \Phi(t,r)\Phi(r,t-j)b_{t-1-j} \\ &= \Phi(t,s)x_s + \sum_{j=0}^{t-1-s} \Phi(t,r)\Phi^{-1}(t-j,r)b_{t-1-j} \\ &= \Phi(t,s)x_s + \sum_{j=0}^{t-1-s} \Gamma(t,t-j)b_{t-1-j} \end{aligned}$$

where $r \in \mathbb{Z}$ is arbitrary. Assuming that all eigenvalues of A are strictly smaller than one in absolute, i.e. $\rho(A) < 1$, the term $\Phi(t,s)x_s = A^{t-s}x_s$ converges to zero for $s \to -\infty$. Hence, the particular solution becomes

$$x_t^{(p)} = \sum_{j=0}^{\infty} \Phi(t,t-j)b_{t-1-j} = \sum_{j=0}^{\infty} \Gamma(t,t-j)b_{t-1-j}.$$

Noting that $\Phi(t,s) = A^{t-s}$ and $\Gamma(t,s) = A^{t-s}$, this result exactly matches the backward solution (3.15).

A similar argument holds with respect to the forward solution. Because $c_t = (c_t - c_{t+1}) + (c_{t+1} - c_{t+2}) + \cdots + (c_{s-1} - c_s) + c_s$, for $t > s$,

$$c_t = c_s - \sum_{j=0}^{s-t-1} \Phi^{-1}(t+1+j)b_{t+j}$$

Hence, multiplying by $\Phi(t)$

$$x_t = \Phi(t)c_t = \Phi(t)c_s - \Phi(t) \sum_{j=0}^{s-t-1} \Phi^{-1}(t+1+j)b_{t+j}.$$

Because $x_s = \Phi(s)c_s$,

$$\begin{aligned} x_t &= \Phi(t)\Phi^{-1}(s)x_s - \sum_{j=0}^{s-t-1} \Phi(t)\Phi^{-1}(r)\Phi(r)\Phi^{-1}(t+1+j)b_{t+j} \\ &= \Phi(t,s)x_s - \sum_{j=0}^{s-t-1} \Phi(t,r)\Phi(r,t+1+j)b_{t+j} \\ &= \Phi(t,s)x_s - \sum_{j=0}^{s-t-1} \Phi(t,r)\Phi^{-1}(t+1+j,r)b_{t+j} \\ &= \Phi(t,s)x_s - \sum_{j=0}^{s-t-1} \Gamma(t,t+1+j)b_{t+j} \end{aligned}$$

where $r \in \mathbb{Z}$ is arbitrary. Assuming that all eigenvalues of A are strictly greater than one in absolute, i.e. $\rho(A) > 1$, the term $\Phi(t,s)x_s = A^{t-s}x_s$ converges to zero for $s \to \infty$. Hence, the particular solution becomes

$$x_t^{(p)} = -\sum_{j=0}^{\infty} \Phi(t,t+1+j)b_{t+j} = -\sum_{j=0}^{\infty} \Gamma(t,t+1+j)b_{t+j}.$$

Noting that $\Phi(t,s) = A^{t-s}$ and $\Gamma(t,s) = A^{t-s}$, this results exactly in the forward solution (3.16).

3.3 Two-dimensional Systems

Many theoretical economic models are reduced and investigated as two-dimensional systems. Thus, we devote this section to the analysis of such systems. They also encompass all qualitatively possible solutions. In case the system is of dimension $d = 2$ a necessary and sufficient condition for asymptotic stability is given in the following theorem.

Theorem 3.4. The homogeneous two-dimensional system has an asymptotically stable solution if and only if

$$|\mathrm{tr}(A)| < 1 + \det A < 2. \tag{3.18}$$

Proof. The characteristic polynomial, $\mathcal{P}(\lambda)$, of a 2×2 matrix is $\mathcal{P}(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det A$. The roots of this quadratic polynomial are thus $\lambda_{1,2} = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}^2(A) - 4\det A}}{2}$. Suppose that the zero point is asymptotically stable then

$|\lambda_{1,2}| < 1$. But, in the case of *real roots*, this is equivalent to the following two inequalities:

$$\begin{aligned} -2 - \mathrm{tr}(A) &< \sqrt{\mathrm{tr}^2(A) - 4\det A} < 2 - \mathrm{tr}(A) \\ -2 - \mathrm{tr}(A) &< -\sqrt{\mathrm{tr}^2(A) - 4\det A} < 2 - \mathrm{tr}(A). \end{aligned}$$

Squaring the second inequality in the first line and simplifying gives: $\mathrm{tr}(A) < 1 + \det A$. Squaring the first inequality in the second line gives $-1 - \det A < \mathrm{tr}(A)$. Combining both results gives the first part of the stability condition (3.18). The second part follows from the observation that $\det A = \lambda_1\lambda_2$ and the assumption that $|\lambda_{1,2}| < 1$. If the roots are *complex*, they are conjugate complex, so that the second part of the stability (3.18) results from $\det A = \lambda_1\lambda_2 = |\lambda_1|\,|\lambda_2| < 1$. The first part follows from $\mathrm{tr}^2(A) - 4\det A < 0$ which is equivalent to $0 < \mathrm{tr}^2(A) < 4\det A$. This can be used to show that

$$4\left(1 + \det A - \mathrm{tr}(A)\right) > 4 + \mathrm{tr}^2(A) - 4\mathrm{tr}(A) = \left(2 - \mathrm{tr}(A)\right)^2 > 0.$$

which is the required inequality.

Conversely, if the stability condition (3.18) is satisfied and if the roots are real, we have

$$\begin{aligned} -1 < \frac{-2 + \sqrt{\mathrm{tr}^2(A) - 4\det A}}{2} &< \lambda_1 = \frac{\mathrm{tr}(A) + \sqrt{\mathrm{tr}^2(A) - 4\det A}}{2} \\ &< \frac{\mathrm{tr}(A) + \sqrt{\mathrm{tr}^2(A) + 4 - 4\mathrm{tr}(A)}}{2} \\ &= \frac{\mathrm{tr}(A) + \sqrt{\left(2 - \mathrm{tr}(A)\right)^2}}{2} < 1. \end{aligned}$$

Similarly, for λ_2. If the roots are complex, they are conjugate complex and we have $|\lambda_1|^2 = |\lambda_2|^2 = \lambda_1\lambda_2 = \frac{\mathrm{tr}^2(A) - \mathrm{tr}^2(A) + 4\det A}{4} = \det A < 1$. This completes the proof. □

The inequalities (3.18) are visualized in Figure 3.1 which is analogous to the Figure 2.6 corresponding to the univariate difference equation of order two. The two inequalties are satisfied when $\mathrm{tr}(A)$ and $\det A$ fall within the triangle with vertices $(2, 1), (0, -1), (-2, 1)$.

Phase Diagrams of Two-Dimensional Systems

An additional advantage of two-dimensional systems are that their qualitative properties can be easily visualized by a *phase diagram*. This allows

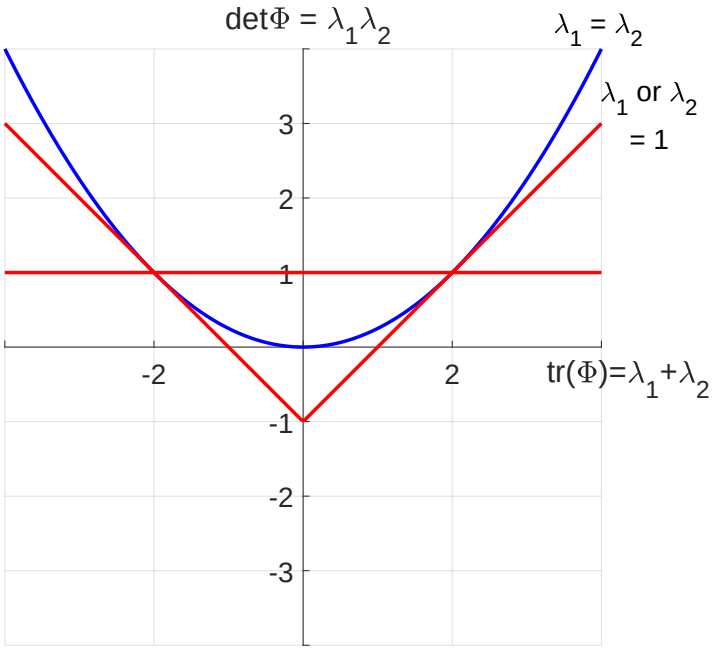


Figure 3.1: Stability Properties of Two-Dimensional Systems

to classify the behavior of two-dimensional linear systems and all nonlinear system which are locally conjugate to it. Consider for this purpose the homogeneous first order system (3.5) written as a two equation system:[8]

$$x_{t+1} = \begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = Ax_t = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}; \tag{3.19}$$

or equivalently

$$x_{1,t+1} = a_{11}x_{1,t} + a_{12}x_{2,t} \tag{3.20}$$
$$x_{2,t+1} = a_{21}x_{1,t} + a_{22}x_{2,t}. \tag{3.21}$$

It is clear that $(0,0)'$ is an equilibrium point for this system. In order to understand the dynamics of the system, we can draw two lines in the (x_1, x_2)-diagram. The first line is given by all points such that the first variable does not change, i.e. where $x_{1,t+1} = x_{1,t}$. From equation (3.20), these points are represented by a line with equation $(a_{11} - 1)x_{1,t} + a_{12}x_{2,t} = 0$. Similarly, the points where the second variable does not change is, from equation (3.21), the line with equation $a_{21}x_{1,t} + (a_{22} - 1)x_{2,t} = 0$. These two lines divide the $\mathbb{R} \times \mathbb{R}$-plane into four regions I, II, III, and IV as in figure 3.2. In this example both lines have positive slopes, but this is not necessarily so.

The dynamics of the system in each of the four regions can be figured out from the signs of the coefficients as follows. Suppose we start at a point on the $x_{1,t+1} - x_{1,t} = (a_{11} - 1)x_{1,t} + a_{12}x_{2,t} = 0$ schedule then we know that the first variable does not change. Now increase $x_{2,t}$ a little bit, but leave $x_{1,t}$ unchanged. This moves us into region I or IV depending on the sign of a_{12}. If a_{12} is positive, this implies that $x_{1,t+1} - x_{1,t} > 0$ so that the first variable increases. Thus, we know that above the $x_{1,t+1} - x_{1,t} = 0$ line, $x_{1,t}$ increases and that below this line $x_{1,t}$ decreases. We can indicate this result in figure 3.2 by arrows from left to right in regions I and IV and arrows from right to left in regions II and III. If a_{12} is negative, we obtain, of course, the opposite result. Similarly, consider the schedule $x_{2,t+1} - x_{2,t} = a_{21}x_{1,t} + (a_{22} - 1)x_{2,t} = 0$ where the second variable does not change. Consider next an increase in $x_{1,t}$ leaving $x_{2,t}$ unchanged. This moves us into region III or IV. If the coefficient a_{21} is positive, this implies that $x_{2,t+1} - x_{2,t} > 0$ so that the second variable must increase. As before we can infer that below the $x_{2,t+1} - x_{2,t} = 0$ line $x_{2,t}$ increases whereas above this line $x_{2,t}$ decreases. We can again indicate this behavior by arrows: upward arrows in regions III and IV and downward arrows in regions I and II. If the sign of a_{21} is negative, the opposite result is obtained. This type of analysis gives us a

[8] We may also interpret the systems as written in deviations from steady state.

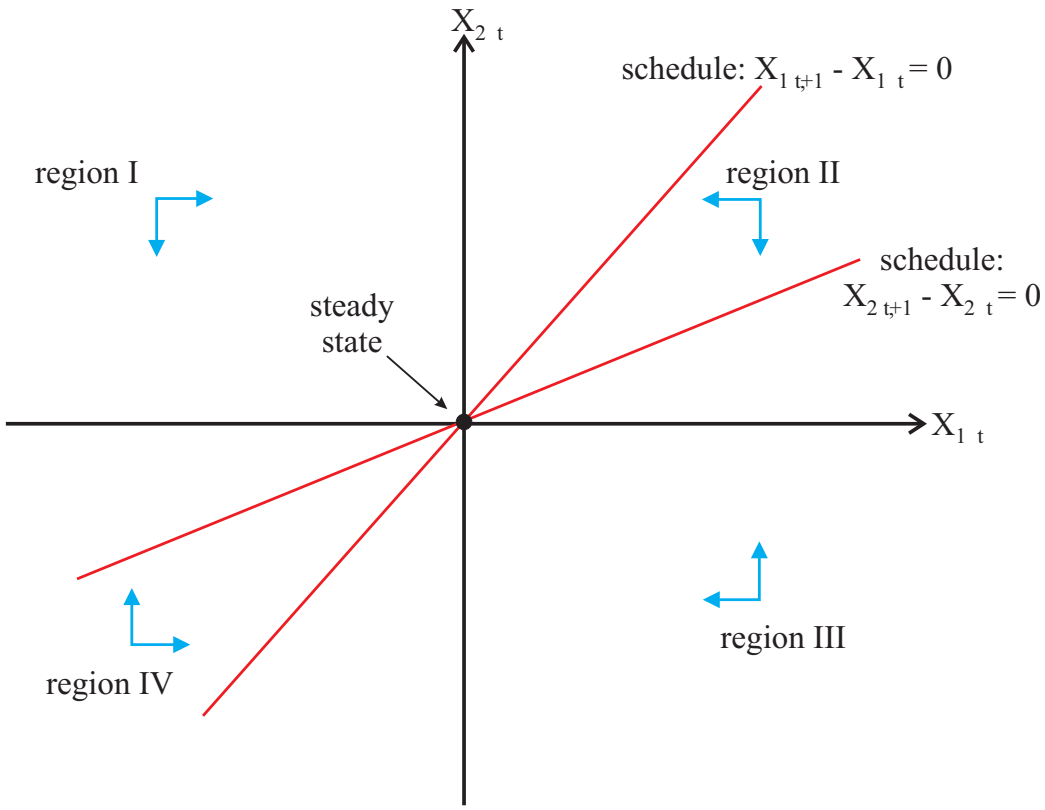


Figure 3.2: Example of a Phase Diagram

phase diagram as in Figure 3.2. This diagram shows us the dynamics of the system starting in the neighborhood of the equilibrium point. In this example, the arrows indicate that, irrespective of the starting point, we will move closer to the steady state. But this corresponds exactly to the definition of an asymptotically stable equilibrium point (see Definition 1.2). Thus, we conclude from this diagram that the steady state is an asymptotically stable equilibrium point.

Of course, the configuration depicted in Figure 3.2 is not the only possible one. In order to classify all qualitatively different configurations, we conjugate the system to a simpler one. If A has Jordan canonical form $A = QJQ^{-1}$ then we make the variable transformation $Y_t = Q^{-1}x_t$. This results in a new first order homogeneous difference equation system:

$$y_{t+1} = Q^{-1}x_{t+1} = Q^{-1}Ax_t = Q^{-1}AQQ^{-1}x_t = Jy_t \tag{3.22}$$

where J has one of the following three forms:

$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ distinct or repeated semisimple real eigenvalues

$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ repeated eigenvalue with only one independent eigenvector

$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ complex eigenvalues: $\lambda_{1,2} = \alpha \pm \imath\beta$

Note that the steady state is not affected by this variable transformation. It is still the point $(0, 0)'$. Let us treat these three cases separately.

case 1: ***distinct or repeated semisimple real eigenvalues*** The variable transformation has effectively decoupled the two-dimensional system into two separate homogeneous first order difference equations with solutions: $y_{1,t} = \lambda_1^t y_{1,0}$ and $y_{2,t} = \lambda_2^t y_{2,0}$ where $y_{1,0}$ and $y_{2,0}$ are given initial values. From the previous discussion we know that the steady state is asymptotically stable if and only if both eigenvalues are smaller than one in absolute value. Such a situation is plotted in Figure 3.3. The arrows indicate that for every starting point the system will converge towards the equilibrium point. As an example we have plotted four trajectories starting at the points $(1, 1), (1, -1), (-1, 1), \text{and}(-1, -1)$, respectively.

Figure 3.4 displays a situation where the equilibrium point is unstable. Indeed both eigenvalues are larger than one and the trajectories quickly

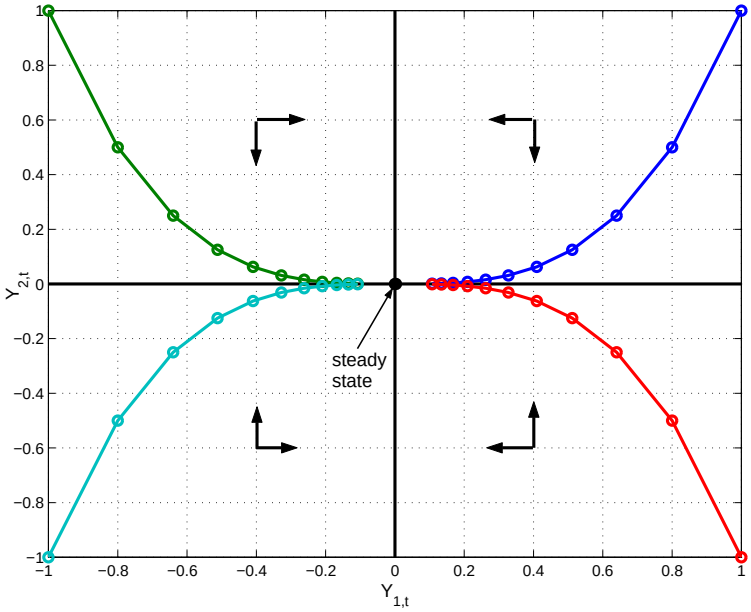


Figure 3.3: Asymptotically Stable Steady State ($\lambda_1 = 0.8, \lambda_2 = 0.5$)

diverge in the directions indicated by the arrows. In the Figure, we have plotted again four trajectories with the same starting values as in the previous example. Only, if both initial values $y_{1,0}$ and $y_{2,0}$ are equal to zero, will the system not diverge and remain bounded. It will actually be stuck at the equilibrium point. An economic example of this configuration will be discussed in Section 4.4.

Figure 3.5 shows an interesting configuration which is often encountered in economic models, especially in those which involve rational expectations (see Sections 4.1 and 4.2 for examples). We have one eigenvalue smaller than one in absolute value and one eigenvalue larger than one in absolute value, i.e. $|\lambda_1| > 1 > |\lambda_2|$. This implies that the system is expanding in the direction of the eigenvector corresponding to λ_1, but is contracting in the direction of the eigenvector corresponding to λ_2. This configuration of the eigenvalues leads to a *saddle point* equilibrium (see Definition 1.5). Although the steady state is unstable, as almost all trajectories diverge, there are some initial values for which the system converges to the steady state. In Figure 3.5 all trajectories starting on the y-axis converge to the steady state. Thus, given an initial value y_{20} for $Y_{2,t}$, the requirement that the solution must be bounded uniquely determines an initial condition for $y_{1,t}$ too, which in this reduced setting is just $y_{10} = 0$. Thus, the solution is given by $y_{1,t} = 0$ and $y_{2,t} = \lambda_2^t y_{2,0}$

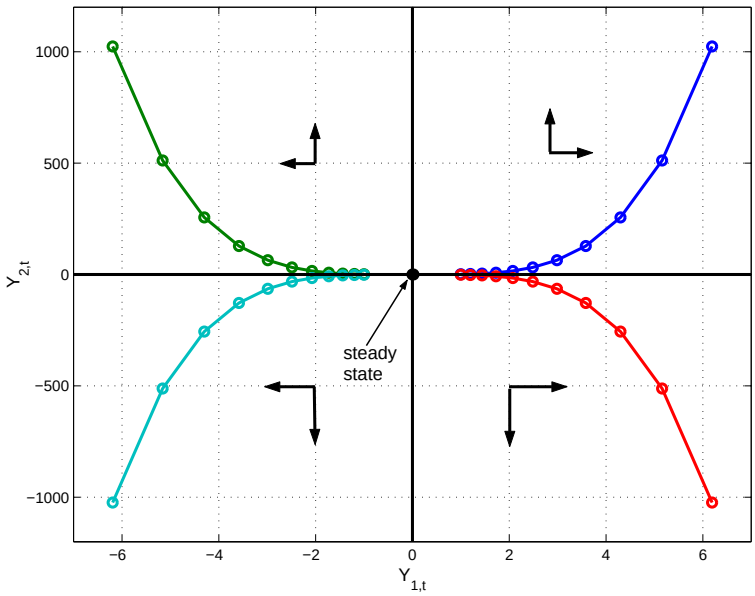


Figure 3.4: Unstable Steady State ($\lambda_1 = 1.2, \lambda_2 = 2$)

for some initial value $y_{2,0}$. Note that the saddle path, in contrast to the other paths, is a straight line through the origin. This property is carried over when the system is transformed back to its original variables. In fact, the solution becomes $x_{1,t} = q_{12}\lambda_2^t y_{2,0}$ and $x_{2,t} = q_{22}\lambda_2^t y_{2,0}$ where $(q_{12}, q_{22})'$ is the eigenvector corresponding to λ_2. Thus, the ratio of $x_{1,t}$ and $x_{2,t}$ equals q_{12}/q_{22} constant.[9] As saddle point equilibria are very prominent in economics, we investigate this case in depth in Section 3.4. In particular, we will go beyond the two-dimensional systems and analyze the role of initial values in detail.

When there are multiple eigenvalues with two independent eigenvectors, A can again be reduced by a similarity transformation to a diagonal matrix. The trajectories are then straight lines leading to the origin if the eigenvalue is smaller than one as in Figure 3.6, and straight lines leading away from the origin if the eigenvalue is larger than one in absolute terms.

When one eigenvalue equals one whereas the second eigenvalue is smaller than one in absolute value, the system violates the hyperbolicity assumption and a degenerate situation emerges. Whereas $Y_{1,t}$ remains at its starting value y_{10}, $Y_{2,t}$ converges to zero so that the system converges

[9] If $q_{22} = 0$, we take the ratio $x_{2,t}/x_{1,t}$ which again defines a straight line through the origin.

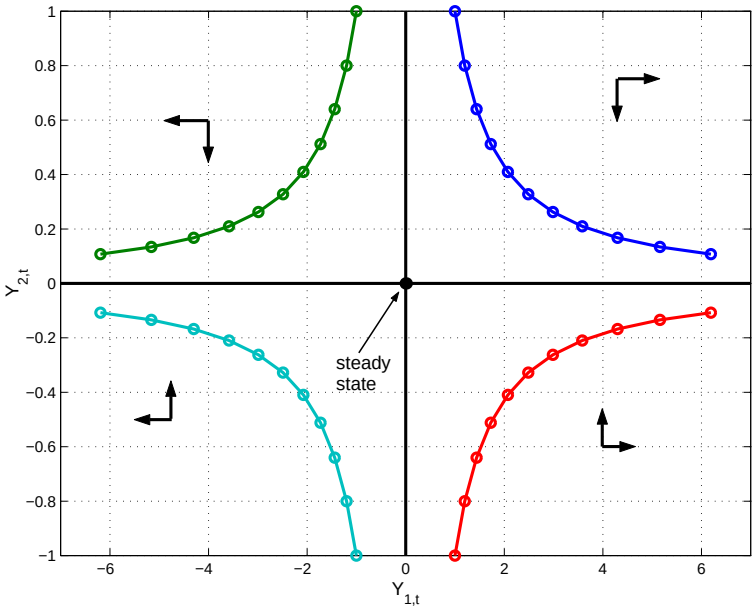


Figure 3.5: Saddle Point Steady State ($\lambda_1 = 1.2, \lambda_2 = 0.8$)

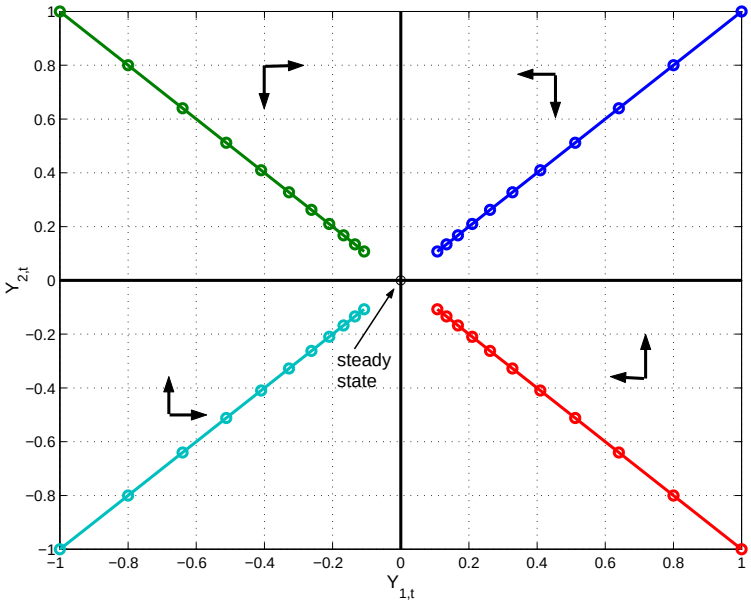


Figure 3.6: Repeated Roots with Asymptotically Stable Steady State ($\lambda_1 = \lambda_2 = 0.8$)

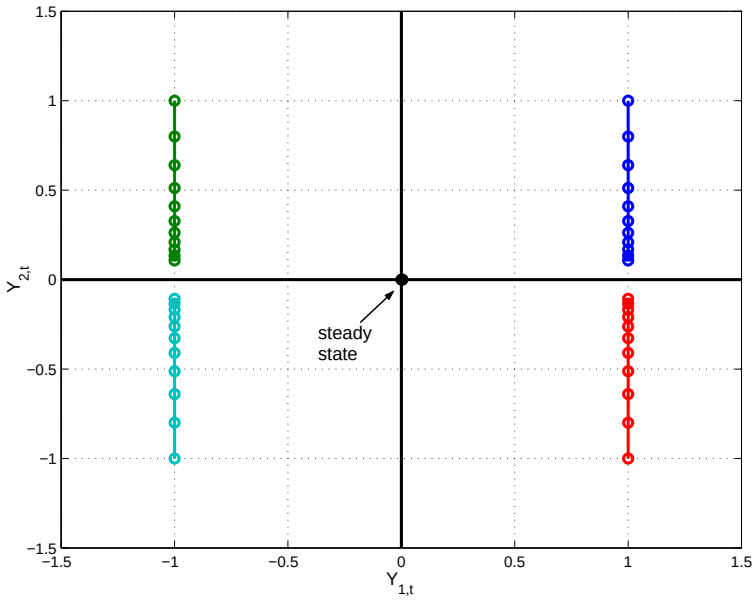


Figure 3.7: Degenerate Steady State ($\lambda_1 = 1, \lambda_2 = 0.8$)

to $(y_{10}, 0)'$ as is exemplified by figure 3.7. Only when $y_{10} = 0$ will there be a convergence to the steady state $(0, 0)$. However, $(0, 0)$ is not the only steady state because the definition of an eigenvalue implies that $A - I$ is singular. Thus, there exists $x^* \neq 0$ such that $x^* = Ax^*$.

In case that the first eigenvalue equals minus one, there is no convergence as $y_{1,t}$ will oscillate between y_{10} and $-y_{10}$.

case 2: ***repeated eigenvalues with one independent eigenvector*** In this case A can no longer be reduced to a diagonal matrix by a similarity transformation. As J is no longer a diagonal matrix, the locus of all points where $y_{1,t}$ does not change is no longer the x-axis, but is given by the line with equation $(\lambda - 1)y_{1,t} + y_{2,t} = 0$. Figure 3.8 displays this case with an eigenvalue of 0.8 which implies an asymptotically stable steady state. Note that, given our four starting points, the system moves first away from the equilibrium point until it hits the schedule where $Y_{1,t}$ does not change, then it changes direction and runs into the steady state.

case 3: ***complex eigenvalues*** If the two conjugate complex eigenvalues are $\lambda_{1,2} = \alpha \pm \imath\beta$ then A is similar to the matrix

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = |\lambda| \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}.$$

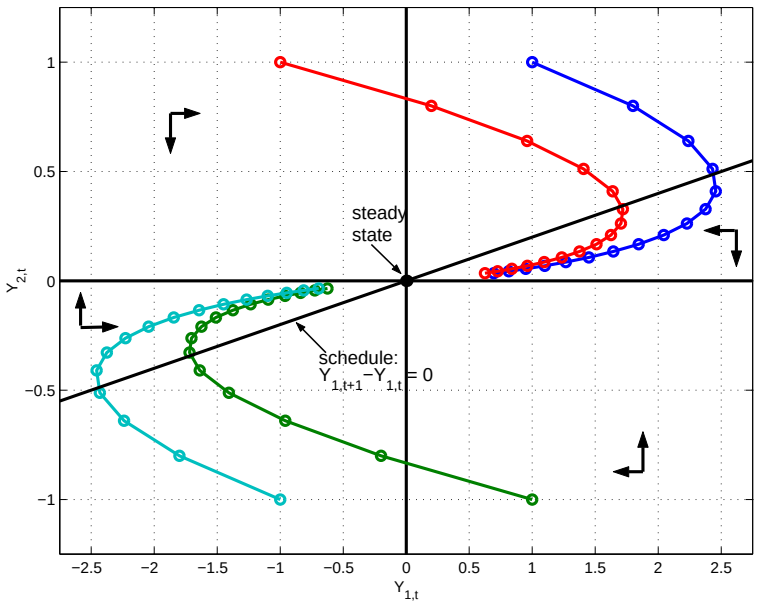


Figure 3.8: Repeated eigenvalues one independent eigenvector ($\lambda = 0.8$)

where $|\lambda| = \sqrt{\alpha^2 + \beta^2}$ and $\omega = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$. Figure 3.9 shows the dynamics in the case of eigenvalues inside the unit circle. One can clearly discern the oscillatory behavior and the convergence to the steady state. Figure 3.10 displays a situation with an unstable steady state where all trajectories move away from the steady state. Finally Figure 3.11 displays a degenerate case where the eigenvalues are on the unit circle. In such a situation the system moves around its steady state in a circle. The starting values are $(0.25, 0.25)$, $(0.5, 0.5)$, $(1, 1)$, and $(1.5, 1.5)$.

3.4 Boundary Value Problems under Rational Expectations

In this section we discuss the general boundary value problem under rational expectations. In these models there are, typically, not enough initial values to pin down a unique solution. Thus, one has to resort to additional restrictions. These restrictions come from the assumption that the solution must remain bounded. In the well-behaved scenario, this will give just enough additional initial values to determine a unique solution and the model is said to be determinate (see also Section 1.1). Geometrically, this solution has the form of a saddle path and the steady state is a saddle point.

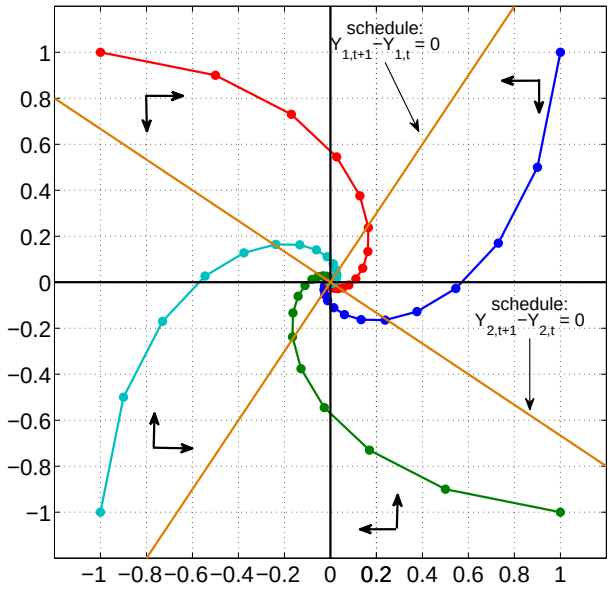


Figure 3.9: Complex eigenvalues with Stable Steady State ($\lambda_{1,2} = 0.7 \pm 0.2i$)

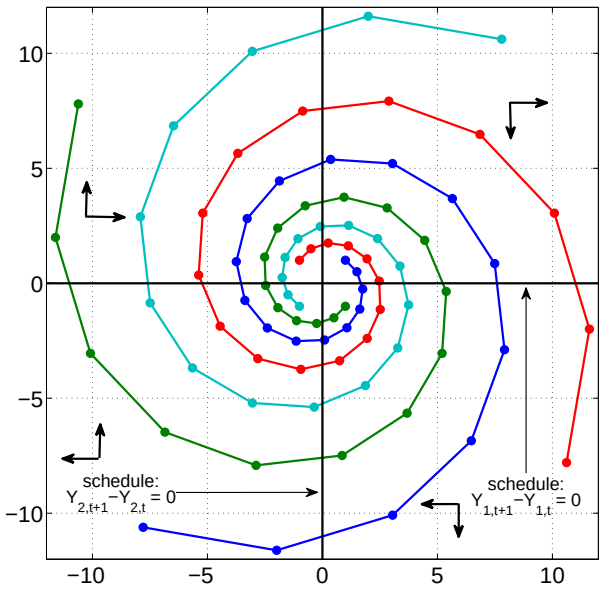


Figure 3.10: Complex eigenvalues with Unstable Steady State ($\lambda_{1,2} = 1 \pm 0.5i$)

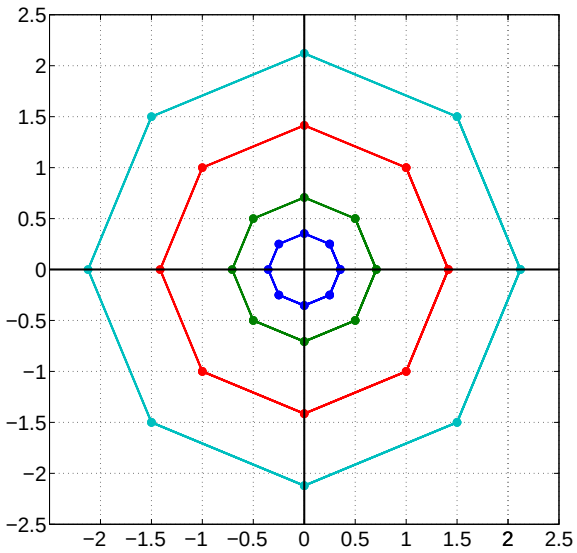


Figure 3.11: Complex eigenvalues on the unit circle ($\lambda_{1,2} = \cos(\pi/4) \pm \imath \sin(\pi/4)$)

A concise treatment of such models in the context of stochastic difference equations was first given by Blanchard and Kahn (1980). Klein (2000) and Sims (2001) provide further insights and solution approaches (see Chapter 5 for further details). As the stochastic setting delivers similar conclusions with respect to uniqueness, we adopt the Blanchard-Kahn setup to the deterministic case by replacing rational expectations by perfect foresight. This framework delivers first order affine nonautonomous difference equations of the form:

$$x_t = Ax_{t-1} + b_t, \qquad t \in \mathbb{Z}. \tag{3.23}$$

Throughout this section we make the following assumptions:

(i) A is a $d \times d$ invertible real matrix, i.e. $A \in \mathbb{GL}(d)$.

(ii) A is hyperbolic, i.e. A has no eigenvalues on the unit circle. Thus, we can view the system (3.23) as the linearized version of a nonlinear one.

(iii) b_t is bounded.

In addition, we are given r initial values with $0 < r < d$ for period 0: [10]

$$c^1 = Rx_0 \tag{3.24}$$

where c^1 is a given r-vector and R is a $(r \times d)$-matrix of rank r. The simplest case is the one where initial values are given for the first r variables:

$$x_{10} = c_1^1, \ldots, x_{r0} = c_r^1,$$

or in matrix form

$$c^1 = \begin{pmatrix} c_1^1 \\ \vdots \\ c_r^1 \end{pmatrix} = \begin{pmatrix} 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} x_{10} \\ \vdots \\ x_{r0} \\ x_{r+1,0} \\ \vdots \\ x_{d0} \end{pmatrix} = (I_r, 0_{r \times d-r}) x_0$$

If $r < n$, these initial conditions are not sufficient to pin a unique solution. There are still $d - r$ degrees of freedom left. We therefore use as an additional requirement that solutions have to be bounded. Thus, a rational expectations model is a *boundary value problem* which consists of a difference equation (3.23), an initial value condition (3.24), and a boundary condition.

The application of the superposition principle expresses the solution of the difference equation (3.23) as the sum of the general solution of the homogeneous equation $x_t^{(g)}$ and a particular solution of the nonhomogeneous equation $x_t^{(p)}$:

$$\begin{aligned} x_t &= x_t^{(g)} + x_t^{(p)} \\ &= Q\Lambda^t Q^{-1} c + x_t^{(p)} \end{aligned}$$

where the d-vector c is yet to be determined. Taking $t = 0$, $x_0 = c + x_0^{(p)}$ or $c = x_0 - x_0^{(p)}$. Given a particular solution, the initial values given by equation (3.24) thus determine c only up to $d - r$ degrees of freedom so that we are lacking $d - r$ additional boundary conditions.

In order to solve the boundary value problem, we partition the Jordan form of A according to the moduli of the eigenvalues. Let $A = Q^{-1}JQ$ where J is the Jordan form of A and where the columns of Q consist of the

[10] The case $r = 0$ can be treated in a similar manner (See the example in Section 4.4). If $r = d$ there are just enough initial conditions and the solution procedure of Section 3.2 can be directly applied.

corresponding eigenvectors (generalized eigenvectors). Define the number of possibly multiple eigenvalues strictly smaller than one by d^s and the number of eigenvalues strictly larger than one by d^u so that $d = d^s + d^u$. Assume that the eigenvalues are ordered in terms of their moduli, then define the matrices Λ_1 and Λ_2 as follows:

$$\Lambda_1 = \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \Lambda_2 = \begin{pmatrix} 0 & 0 \\ 0 & J_2 \end{pmatrix}$$

where J_1 consists of the Jordan segments corresponding to the eigenvalues smaller than one and J_2 to the segments corresponding to the eigenvalues greater than one. We focus on the case where the zero solution is a saddle point, meaning that there exists at least two eigenvalues λ_1 and λ_2 such that $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (see Definition 1.5).[11] With this notation, we propose the following particular solution:

$$x_t^{(p)} = \sum_{j=0}^{\infty} Q\Lambda_1^j Q^{-1} b_{t-j} - \sum_{j=1}^{\infty} Q\Lambda_2^{-j} Q^{-1} b_{t+j} \tag{3.25}$$

The reader is invited to verify that this is indeed a solution to equation (3.23). The solution proposed in equation (3.25) has the property that "variables" corresponding to eigenvalues smaller than one are iterated backwards whereas those corresponding to eigenvalues larger than one are iterated forwards. This ensures that $\{x_t^{(p)}\}$ remains bounded whenever $\{b_t\}$ is. The general solution therefore is of the form

$$x_t = Q\Lambda_1^t Q^{-1} c + Q\Lambda_2^t Q^{-1} c + \sum_{j=0}^{\infty} Q\Lambda_1^j Q^{-1} b_{t-j} - \sum_{j=1}^{\infty} Q\Lambda_2^{-j} Q^{-1} b_{t+j}. \tag{3.26}$$

The Jordan decomposition of A gives rise to a splitting of $\mathbb{R}^d$ into the direct sum of two subspaces $\mathbb{L}^s$ and $\mathbb{L}^u$, called the stable and the unstable subspace, i.e. $\mathbb{R}^d = \mathbb{L}^s \oplus \mathbb{L}^u$. Thereby $\mathbb{L}^s$ is spanned by the (generalized) eigenvectors corresponding to eigenvalues strictly smaller than one and has therefore dimension $d^s = \dim \mathbb{L}^s$ whereas $\mathbb{L}^u$ is spanned by the (generalized) eigenvectors corresponding to eigenvalues strictly larger than one and has dimension $d^u = \dim \mathbb{L}^u$. Given the ordering of the eigenvalues from before $\mathbb{L}^s$ is spanned by the first d^s columns of Q whereas $\mathbb{L}^u$ is spanned by the last d^u columns of Q. Denote the projector of $\mathbb{R}^d$ onto $\mathbb{L}^s$ along $\mathbb{L}^u$ by π^s

[11] If d^s or d^u equal zeros then the terms corresponding to the matrices Λ_1 and Λ_2 are omitted.

and the projector of $\mathbb{R}^d$ onto $\mathbb{L}^u$ along $\mathbb{L}^s$ by π^u, then the solution given in equation (3.26) may be rewritten as

$$\begin{aligned} x_t &= QJ^tQ^{-1}c + \sum_{j=0}^{\infty} QJ^jQ^{-1}\pi^s b_{t-j} - \sum_{j=1}^{\infty} QJ^{-j}Q^{-1}\pi^u b_{t+j} \\ &= A^t c + \sum_{j=0}^{\infty} A^j \pi^s b_{t-j} - \sum_{j=1}^{\infty} A^{-j}\pi^u b_{t+j}. \end{aligned}$$

The equivalence between the two representations of the solution is evident by noting (see f.e. Meyer, 2000, section 5.9) that

$$\pi^s = Q \begin{pmatrix} I_{d_1} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1} \qquad \pi^u = Q \begin{pmatrix} 0 & 0 \\ 0 & I_{d_2} \end{pmatrix} Q^{-1}.$$

The final step consists in finding the constant c. There are two types of restrictions: the first one are the initial values given by equation (3.24); the second one are the requirement that we are only interested in non-exploding solutions. Whereas the first type delivers r restrictions because $\operatorname{rank}(R) = r$, the second type delivers $d_2 = d - r$ restrictions. Thus, c must be determined according to the following equation system:

$$\begin{array}{ll} \textit{initial values:} & R\,c = Rx_0 - R\,x_0^{(p)} = c^1 - R\,x_0^{(p)} \\ \textit{no explosive solutions:} & Q^{(2)}\,c = Q^{(21)}c_1 + Q^{(22)}c_2 = 0 \end{array} \tag{3.27}$$

where $Q^{-1} = \begin{pmatrix} Q^{(11)} & Q^{(12)} \\ Q^{(21)} & Q^{(22)} \end{pmatrix}$, $Q^{(2)} = \begin{pmatrix} Q^{(21)} \vdots Q^{(22)} \end{pmatrix}$ and $c = (c_1', c_2')'$ and where the partitioning of Q^{-1} and c conforms to the partitioning of the eigenvalues. Note that $Q^{(2)}$ is a $d^u \times d$ matrix. Depending on whether the number of independent restrictions is greater, smaller or equal to d, several situations arise.

Theorem 3.5 (Blanchard-Kahn)**.** Let $A \in \mathbb{GL}(n)$ be hyperbolic then the nonhomogeneous difference equation (3.23) with initial values given by (3.24) has a unique nonexplosive solution if and only if

$$\operatorname{rank} \begin{pmatrix} R \\ Q^{(2)} \end{pmatrix} = d. \tag{3.28}$$

The solution is given by equation (3.26) with c uniquely determined from (3.27).

A necessary condition is that $r = \operatorname{rank}(R) = d^s$: the number of eigenvalues smaller than one, d_1, is equal to the number of independent initial conditions, $r = \operatorname{rank}(R)$.

In many instances the values of the first r variables at given time, say time zero, are given. This is the case when there are exactly r predetermined variables in the system. The initial values then fix the first r values of c: $c_1 = x_{10} - x_{10}^{(p)}$ where x_{10} and $x_{10}^{(p)}$ denote the first r values of x_0 and $x_0^{(p)}$.

Corollary 3.1. If $R = (I_r, 0_{r\times(d-r)})$ with $r = \text{rank}(R) = d^s$, the necessary and sufficient condition (3.28) reduces to $Q^{(22)}$ is invertible. Given that c_1 is fixed by the initial conditions, $c_2 = -(Q^{(22)})^{-1}Q^{(21)}c_1$.

If $k = \text{rank}(R) > d^s$ then there are too many restrictions and there is no nonexplosive solution. We may, however, soften the boundedness condition and accept explosive solutions in this situation.

If $r = \text{rank}(R) < d^s$ there are not enough initial conditions so that it is not possible to pin down c uniquely. We thus have an infinite amount of solutions and we call such a situation *indeterminate*. The multiplicity of equilibria or indeterminacy opens up the possibility of *sunspot* equilibria.[12] Sunspot equilibria have been introduced by Cass and Shell (1983), Azariadis (1981), and Azariadis and Guesnerie (1986) (see also Azariadis (1993) and Farmer (1993)).

3.5 Nonnormal Difference Equations

[12]Sunspot equilibria explore the idea that extraneous beliefs about the state of nature influence economic activity. The disturbing feature of sunspot equilibria is that economic activity may change across states although nothing fundamental has changed.

Chapter 4

Examples of Linear Deterministic Systems of Difference Equations

4.1 Exchange Rate Overshooting

4.1.1 Introduction

A classic example for a system with one predetermined and one so-called "jump"-variable is the exchange rate overshooting model by Dornbusch (Dornbusch (1976)).[1] The model describes the behavior of the price level and the exchange rate in a small open economy. It consists of an IS equation, a price adjustment equation, and a LM equation. In addition, the uncovered interest rate parity (UIP) is assumed together with rational expectations:

$$\begin{aligned} y_t^d &= \delta(e_t + p^* - p_t) - \sigma(r_t - p_{t+1} + p_t), && \text{(IS)} \\ p_{t+1} - p_t &= \alpha(y_t^d - y), && \text{(price adjustment)} \\ m - p_t &= \phi y - \lambda r_t, && \text{(LM)} \\ r_t &= r^* + e_{t+1} - e_t, && \text{(UIP)} \end{aligned}$$

where the parameters $\delta, \sigma, \alpha, \phi$, and λ are all positive. The variables are all expressed in logarithms. The IS-equation represents the dependence of aggregate demand y_t^d on the relative price of foreign to home goods and on the real interest rate. A devaluation of the exchange rate[2], an increase in the

[1] Rogoff (Rogoff (2002)) provides an appraisal of this influential paper.

[2] The exchange rate e_t is quoted as the price of a unit of foreign currency in terms of the domestic currency. An increase in e_t therefore corresponds to a devaluation of the home currency.

foreign price level, p^*, or a decrease in the domestic price level p_t all leads to an increase in aggregate demand. On the other hand, an increase in the domestic nominal interest rate, r_t, or a decrease in the expected inflation rate, $p_{t+1}-p_t$, lead to a reduction in aggregate demand. A crucial feature of the Dornbusch model is that prices are sticky and adjust only slowly. In particular, the price adjustment $p_{t+1} - p_t$ is proportional to the deviation of aggregate demand from potential output y. If aggregate demand is higher than potential output, prices increase whereas, if aggregate demand is below potential output, prices decrease. In particular, the price level is treated as predetermined variable whose value is fixed in the current period. The LM-equation represents the equilibrium on the money market. The demand for real balances, $m - p_t$, depends positively on potential output[3] and negatively on the domestic nominal interest rate. The model is closed by assuming that the uncovered interest parity holds where r^* denotes the foreign nominal interest rate. In contrast to the price level, the exchange rate is not predetermined. It can immediately adjust within the current period to any shock that may occur. For simplicity, the exogenous variables y, m, r^*, and p^* are assumed to remain constant.

This system can be reduced to a two-dimensional system in the exchange rate and the price level:

$$e_{t+1} - e_t = \frac{1}{\lambda}(\phi y + p_t - m) - r^* \tag{4.1}$$

$$p_{t+1} - p_t = \frac{\alpha}{1-\alpha\sigma}\left[\delta(e_t + p^* - p_t) - y - \frac{\sigma}{\lambda}(\phi y - m + p_t)\right] \tag{4.2}$$

The first equation was obtained by combining the LM-equation with the UIP. The second equation was obtained by inserting the IS-equation into the price adjustment equation, replacing the nominal interest rate using the LM-equation and then solving for $p_{t+1} - p_t$.

The steady state of this system is obtained by setting $e_t = e^{ss}$ and $p_t = p^{ss}$ for all t and solving for this two variables:

$$p^{ss} = \lambda r^* + m - \phi y \tag{4.3}$$

$$e^{ss} = p^{ss} - p^* + \frac{1}{\delta}(y + \sigma r^*) \tag{4.4}$$

In the steady state, UIP and the price adjustment imply $r_t = r^*$ and $y_t^d = y$. The system can be further reduced by writing it in terms of deviations from

[3]This represents a simplification because money demand should depend on aggregate demand, y_t^d, and not on potential output, y. However, we adopt this simplified version for expositional purposes.

steady state:

$$\begin{aligned}\begin{pmatrix} e_{t+1} - e^{ss} \\ p_{t+1} - p^{ss} \end{pmatrix} &= \begin{pmatrix} 1 & \frac{1}{\lambda} \\ \dfrac{\alpha\delta}{1-\alpha\sigma} & 1 - \dfrac{\alpha(\delta + \frac{\sigma}{\lambda})}{1-\alpha\sigma} \end{pmatrix} \begin{pmatrix} e_t - e^{ss} \\ p_t - p^{ss} \end{pmatrix} \\ &= \Phi \begin{pmatrix} e_t - e^{ss} \\ p_t - p^{ss} \end{pmatrix} \end{aligned} \tag{4.5}$$

4.1.2 Analysis of the Dynamic Properties

The dynamic behavior of the system (4.5) depends on the eigenvalues of Φ. Denote the characteristic polynomial of Φ by $\mathcal{P}(\mu)$ and the two corresponding eigenvalues by μ_1 and μ_2, then

$$\mathcal{P}(\mu) = (\mu - \mu_1)(\mu - \mu_2) = \mu^2 - \mathrm{tr}(\Phi)\mu + \det \Phi$$

Without additional assumptions on the parameters, it is impossible to obtain further insights into the qualitative behavior of the system. We suppose that the price adjustment is sufficiently slow. Specifically, we assume that

$$0 < \alpha\sigma < 1 \quad \text{and} \quad \alpha < \frac{4}{(2 + 1/\lambda)(\delta + 2\sigma)},$$

we obtain:

$$\begin{aligned} \mathrm{tr}\Phi = \mu_1 + \mu_2 &= 2 - \frac{\alpha}{1-\alpha\sigma}\left(\delta + \frac{\sigma}{\lambda}\right) < 2 \\ \det \Phi = \mu_1\mu_2 &= 1 - \frac{\alpha}{1-\alpha\sigma}\left(\delta + \frac{\sigma}{\lambda} + \frac{\delta}{\lambda}\right) < 1 \\ \triangle = (\mathrm{tr}\Phi)^2 - 4\det\Phi &= \left[\frac{\alpha}{1-\alpha\sigma}\left(\delta + \frac{\sigma}{\lambda}\right)\right]^2 + \frac{4\alpha\delta}{\lambda(1-\alpha\sigma)} > 0 \\ \mathcal{P}(1) = (1-\mu_1)(1-\mu_2) &= -\frac{\alpha\delta}{\lambda(1-\alpha\sigma)} < 0 \\ \mathcal{P}(-1) = (1+\mu_1)(1+\mu_2) &= 4 - \frac{\alpha}{1-\alpha\sigma}\left[2\left(\delta + \frac{\sigma}{\lambda}\right) + \frac{\delta}{\lambda}\right] > 0 \end{aligned}$$

where $\triangle$ denotes the discriminant of the quadratic equation. The above inequalities have the following implications for the two eigenvalues:

- $\triangle > 0$ implies that the eigenvalues are real;
- $\mathcal{P}(1) < 0$ implies that they lie on opposite sides of 1;
- $\mathrm{tr}\Phi < 2$ implies that the sum of the eigenvalues is less than 2;

- $\mathcal{P}(-1) > 0$ finally implies that one eigenvalue, say μ_1, is larger than 1 whereas the second eigenvalue, μ_2, lies between -1 and 1.

Because the eigenvalues are distinct, we can diagonalize Φ as $\Phi = Q\Lambda Q^{-1}$ where Λ is a diagonal matrix with μ_1 and μ_2 on its diagonal. The column of the matrix Q consist of the eigenvectors of Φ. Multiplying the system (4.5) by Q^{-1}, we obtain the transformed system:

$$\begin{aligned}\begin{pmatrix}\hat{e}_{t+1}\\ \hat{p}_{t+1}\end{pmatrix} &= Q^{-1}\begin{pmatrix}e_{t+1}-e^{ss}\\ p_{t+1}-p^{ss}\end{pmatrix} = Q^{-1}\Phi Q Q^{-1}\begin{pmatrix}e_t-e^{ss}\\ p_t-p^{ss}\end{pmatrix}\\ &= \begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix}\begin{pmatrix}\hat{e}_t\\ \hat{p}_t\end{pmatrix} \qquad (4.6)\end{aligned}$$

Through this change of variables we have obtained a decoupled system: the original two-dimensional system is decomposed into two one-dimensional homogenous difference equations. These two equations have the general solution:

$$\begin{aligned}\hat{e}_t &= c_1\mu_1^t\\ \hat{p}_t &= c_2\mu_2^t\end{aligned}$$

where c_1 and c_2 are two constants yet to be determined. Transforming the solution of the decoupled systems back into the original variables yields:

$$e_t = e^{ss} + c_1 q_{11}\mu_1^t + c_2 q_{12}\mu_2^t \qquad (4.7)$$

$$p_t = p^{ss} + c_1 q_{21}\mu_1^t + c_2 q_{22}\mu_2^t \qquad (4.8)$$

where $Q = (q_{ij})_{i,j=1,2}$. Because $\mu_1 > 1$ this represents an unstable system. As time evolves the unstable eigenvalue will eventually dominate. In order to avoid this explosive behavior, we set c_1 equal to zero. The second constant c_2 can be determined from the boundary condition associated with the predetermined variable, in our case the price level. Suppose the system starts in period zero and we are given a value p_0 for the price level in this period, then according to equation (4.8) $c_2 = \frac{(p_0 - p^{ss})}{q_{22}}$, provided $q_{22} \neq 0$. Combining all these elements with the equations (4.7) and (4.8) leads to the equation for the saddle path:

$$e_t = e^{ss} + \frac{q_{12}}{q_{22}}\left(p_t - p^{ss}\right) \qquad (4.9)$$

Next, we show that the saddle path is downward sloping, i.e. that $\frac{q_{12}}{q_{22}} < 0$. This can be established by investigating the defining equations for the eigenvector corresponding to the second eigenvalue μ_2. They are given by $(\phi_{11}-\mu_2)q_{12}+\phi_{12}q_{22} = 0$ and $\phi_{21}q_{12}+(\phi_{22}-\mu_2)q_{22} = 0$. Because $(\phi_{11}-\mu_2) > 0$

and $\phi_{12} > 0$, q_{12} and q_{22} must be of opposite sign. The same conclusion is reached using the second equation. This reasoning also shows that $q_{22} \neq 0$. Suppose that $q_{22} = 0$ then q_{12} must also be zero because $(\phi_{11} - \mu_2) > 0$. This, however, contradicts the assumption that $(q_{12}, q_{22})'$ is an eigenvector. Note that although the eigenvector is not uniquely determined, its direction and thus the slope of the saddle path is.

The Dornbusch model is most easily analyzed in terms of a phase diagram representing the price level and the exchange rate as in figure 4.1. The graph consists of two schedules: $p_{t+1} - p_t = 0$ and $e_{t+1} - e_t = 0$. Their intersection determines the steady state denoted by **S**. These two schedules correspond to the equations (4.4) and (4.3). The $e_{t+1} - e_t = 0$ schedule does not depend on the exchange rate and is therefore horizontal intersecting the price axis at p^{ss}. Above this schedule the exchange rate depreciates whereas below this schedule the exchange appreciates, according to equation (4.5). This is indicated by arrows pointing to the right, respectively to the left. The $p_{t+1} - p_t = 0$ schedule is upward sloping. To its left, prices are decreasing whereas to its right prices are increasing, according to equation (4.5). The two schedules divide the e-p-quadrant into four regions: I, II, III, and IV. In each region the movement of e and p is indicated by arrows. In the Dornbusch model the price level is sticky and considered to be a predetermined variable. Suppose that in period 0 its level is given by p_0. The exchange rate in this period is not given, but endogenous and has to be determined by the model. Suppose that the exchange rate in period 0 is at a level corresponding to point A. This point is to the left of the $p_{t+1} - p_t = 0$ schedule and above the $e_{t+1} - e_t = 0$ schedule and therefore in region I. This implies that the price level has to fall and the exchange rate to increase. The path of e and p will continue in this direction until they hit the $e_{t+1} - e_t = 0$ schedule. At this time the system enters region IV and the direction is changed: both the price level and the exchange rate decrease. They will so forever. We are therefore on an unstable path. Consider now an exchange rate in period 0 corresponding to point B. Like A, this point is also in region I so that the exchange rate increases and the price level decreases. However, in contrast to the previous case, the path starting in B will hit the $p_{t+1} - p_t = 0$ schedule and move into region II. In this region, both the price level and the exchange rate increase forever. Again this cannot be a stable path. Thus, there must be an exchange rate smaller than the one corresponding to point B, but higher than the one corresponding to point A, which sets the system on a path leading to the steady steady. This is exactly the exchange rate which corresponds to the saddle path given by equation (4.9). In this way the exchange rate in period 0 is pinned down uniquely by the requirement that the path of $(e_t, p_t)'$ converges.

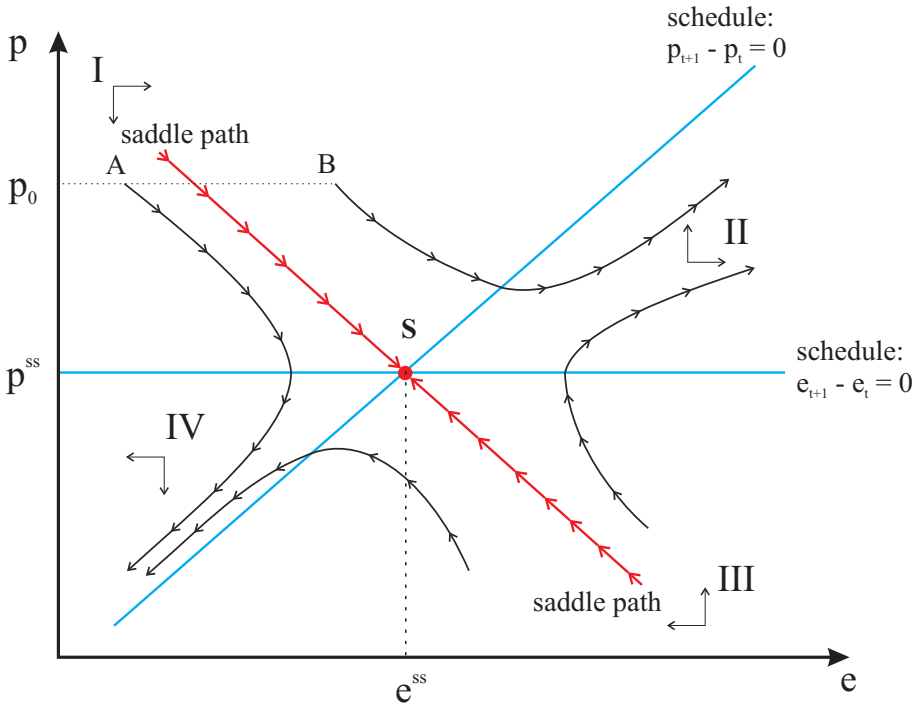


Figure 4.1: Dornbusch's Overshooting Model

4.1.3 Effects of an Increase in Money Supply

The phase diagram is also very convenient in analyzing the effects of changes in the exogenous variables. Consider, for example, an unanticipated permanent increase in money supply. This moves, according to equation (4.3), the $e_{t+1} - e_t = 0$ schedule up and, according to (4.4), the $p_{t+1} - p_t = 0$ to the left as shown in figure 4.2. The steady state therefore jumps from $\mathbf{S}_{old}$ to $\mathbf{S}_{new}$ and the new saddle path goes through $\mathbf{S}_{new}$. Suppose that the price level was initial at p^{ss}_{old}. As the price level cannot react to the new situation it will remain initially at the old steady state level. The exchange rate, however, can adapt immediately and jumps to e_0 such that the system is on the new saddle path. As this value lies typically above the new steady level, we say that the exchange rate overshoots. The reason for this "excess" depreciation of the exchange rate is the stickiness of the price level. In the short-run, the exchange rate carries all the burden of the adjustment. As time evolves the system moves along its saddle path to its new steady state. During this transition the price level increases and the exchange rate appreciates. Thus, the immediate reaction of the economy is a depreciation of the exchange rate coupled with an expected appreciation.

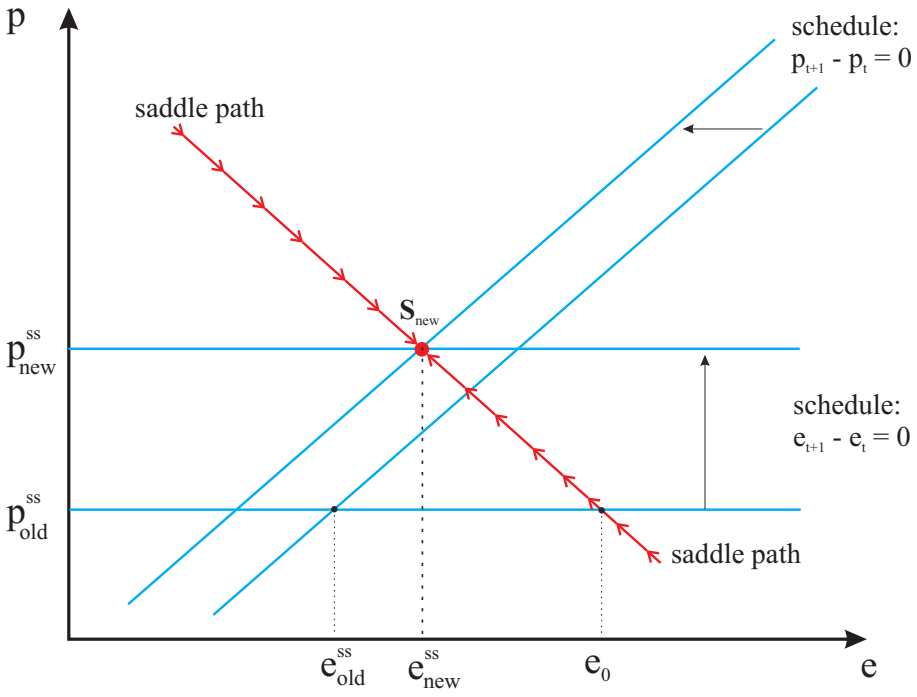


Figure 4.2: Unanticipated Increase in Money Supply Dornbusch's Overshooting Model

4.2 Optimal Growth Model

4.2.1 Introduction

In contrast to the previously discussed Solow model (see section 2.3.2), the optimal growth model seeks to determine the saving-consumption decision optimally.[4] Consider for this purpose a planer who seeks the optimize the discounted utility stream from per capita consumption (c_t). If C_t and L_t denote aggregate consumption and aggregate labor input in period t then per capita consumption is given by $c_t = C_t/L_t$. As before labor input increases at the exogenously given rate $\mu > 0$, i.e. $L_{t+1} = (1+\mu)L_t$. The planner is assumed to maximize the following Bentham type objective function:

$$\begin{aligned} V(c_0, c_1, \cdots) &= \sum_{t=0}^{\infty} \beta^t L_t U(c_t) \\ &= L_0 \sum_{t=0}^{\infty} (\beta(1+\mu))^t U(c_t), \qquad 0 < \beta(1+\mu) < 1. \end{aligned} \tag{4.10}$$

The constant β is called the subjective discount rate. The period utility function $U : \mathbb{R}^+ \to \mathbb{R}$ is continuously differentiable, increasing, strictly concave, and, in order to avoid corner solutions, fulfills $\lim_{c\to 0} U'(c) = \infty$.[5]

The rest of the specification is exactly the same as for the Solow model (see section 2.3.2): Output is produced according to a neoclassical aggregate production satisfying the Inada conditions. Recognizing that investment in period t, I_t equals $I_t = K_{t+1} - (1-\delta)K_t$ the national accounting identity becomes:

$$C_t + I_t = C_t + K_{t+1} - (1-\delta)K_t = F(K_t, L_t)$$

or in per capita terms

$$c_t + (1+\mu)k_{t+1} - (1-\delta)k_t = f(k_t)$$

respectively,

$$c_t + (1+\mu)k_{t+1} = f(k_t) + (1-\delta)k_t = h(k_t). \tag{4.11}$$

[4] The optimal growth model originates in the work of Ramsey (1928). It has been widely analyzed and stands at the heart of the Real Business Cycle approach. An extensive treatment of this model together with additional references can be found in Stokey and Lucas Jr. (1989).

[5] Instead of a Bentham type objective function one could also work with the conventional one: $V(c_0, c_1, \cdots) = \sum_{t=0}^{\infty} \beta^t U(c_t)$ with $0 < \beta < 1$. This modification will, however, not change the qualitative implications of the model.

The first order condition for the optimum is given by the Euler-equation, sometimes also called the Keynes-Ramsey rule:

$$U'(c_t) = \beta h'(k_{t+1})U'(c_{t+1}) \tag{4.12}$$

Thus, the Euler-equation equates the marginal rate of transformation, $1/h'(k_{t+1})$, to the marginal rate of substitution, $\beta U'(c_{t+1})/U'(c_t)$.

The equation system consisting of the transition equation (4.11) and the Euler-equation (4.12) constitutes a nonlinear difference equation system. The analysis of this system proceeds in the usual manner. First, we compute the steady state(s). Then we linearize the system around the steady state. This gives a linear homogeneous difference equation in terms of deviations from steady state. We find the solution of this difference equation using the superposition principle. Finally, we select, if possible, one solution using initial conditions and boundedness arguments.

4.2.2 Steady State

The steady state (k^*, c^*) is found by setting $c^* = c_t = c_{t+1}$ and $k^* = k_t = k_{t+1}$ in equations (4.11) and (4.12). This results in the nonlinear equation system:

$$\Delta k = 0: \qquad c^* = f(k^*) - (\delta + \mu)k^* \tag{4.13}$$

$$\Delta c = 0: \qquad \beta h'(k^*) = \beta(f'(k^*) + 1 - \delta) = 1. \tag{4.14}$$

The $\Delta c = 0$ equation is independent of c and of the shape of the utility function U. This equation therefore determines k^*. The first equation viewed as a function c of k has an inverted U-shape in the (k, c)-plane as can be deducted from the following reasoning:

- $k = 0$ implies $c = 0$. The derivative $\mathrm{d}c/\mathrm{d}k = f'(k) - (\delta + \mu)$ evaluated at $k = 0$ is strictly positive because of the Inada conditions.
- The function reaches a maximum at k^{**} determined by $\mathrm{d}c/\mathrm{d}k = f'(k^{**}) - (\delta + \mu) = 0$. Because $\beta(1 + \mu) < 1$ by assumption, $k^* < k^{**}$. k^{**} is called the *modified golden rule* capital stock. It is larger than the optimal capital stock because of discounting.
- For $k > k^{**}$ the $\Delta k = 0$ schedule declines monotonically and crosses the x-axis at $k^{\max}$. This is the maximal value of capital sustainable in the long-run. It is achieved when the consumption is reduced to zero. Thus, $k^{\max}$ is determined from the equation $f(k^{\max}) = (\delta + \mu)k^{\max}$.

The shape of both schedules is plotted in figure 4.3 as the blue lines. They cross at point **E** which corresponds to the unique nonzero steady state of the system.

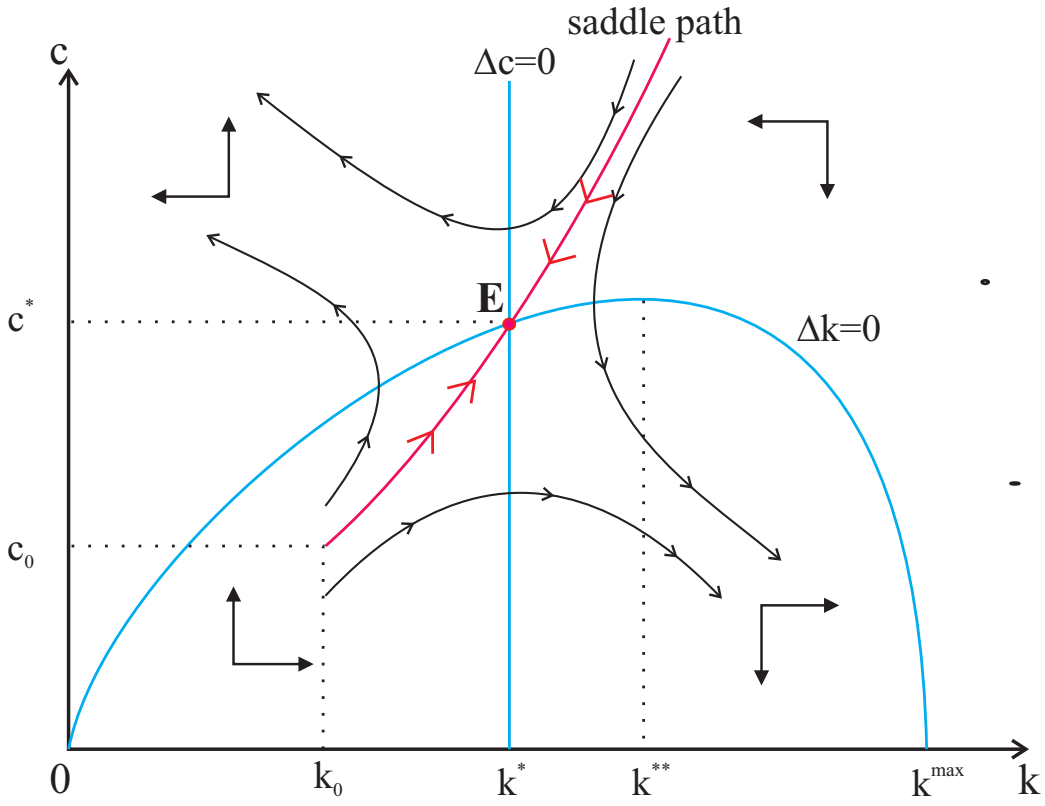


Figure 4.3: Phase diagram of the optimal growth model

4.2.3 Discussion of the Linearized System

The equations (4.11) and (4.12) constitute a two-dimensional system of nonlinear difference equations of order one or, after inserting $h(k_t)-(1+\mu)k_{t+1}$ for c_t and $h(k_{t+1})-(1+\mu)k_{t+2}$ for c_{t+1}, a single nonlinear difference equation of order two. We therefore need two boundary conditions to pin down a solution uniquely. One condition is given by the initial value of the per capita capital stock $k_0 > 0$.

In order to study the dynamics of the system in detail, rewrite the Euler equation (4.12) as

$$U'(c_{t+1}) - U'(c_t) = [1 - \beta h'(k_{t+1})]U'(c_{t+1}). \tag{4.15}$$

From this equation we can deduce that $c_{t+1} \geq c_t$ when $k_{t+1} < k^*$ and vice versa. Thus, to left of the $(\Delta c = 0)$-schedule consumption rises whereas to the right consumption falls. Similarly, the transition equation (4.11) implies that $k_{t+1} \geq k_t$ when c_t is lower than the corresponding c^* implied by the $(\Delta k = 0)$-schedule. Thus, the two schedules divide the nonnegative orthant of the (k, c)-plane in four regions. The dynamics in these four regions is indicated in figure 4.3 by orthogonal arrows. In the region to the left of the $(\Delta c = 0)$-schedule and above the $(\Delta k = 0)$-schedule, i.e. the northwest region, consumption would increase whereas the capital intensity would decrease. This dynamics would continue until the c-axis is hit. When this happens, the economy has no capital left and therefore produces nothing but consumes a positive amount. Such a situation is clearly infeasible. Paths with this property have therefore to be excluded. Starting at k_0, there is, however, one path, the saddle path, where the forces which lead to explosive paths, respectively infeasible paths, just offset each other and lead the economy to the steady state. This is the red line in figure 4.3.

The algebraic analysis requires the linearization of the nonlinear equation system (4.11) and (4.12). This leads to:

$$\begin{pmatrix} 1+\mu & 0 \\ \beta U'(c^*)h''(k^*) & U''(c^*) \end{pmatrix} \begin{pmatrix} k_{t+1}-k^* \\ c_{t+1}-c^* \end{pmatrix} = \begin{pmatrix} \beta^{-1} & -1 \\ 0 & U''(c^*) \end{pmatrix} \begin{pmatrix} k_t-k^* \\ c_t-c^* \end{pmatrix}$$

where we used the fact that $\beta h'(k^*) = 1$. Given that $\mu > 0$ and $U''(c^*) < 0$, the matrix on the left hand side is invertible. This then leads to the following linear first order homogenous system:

$$\begin{aligned} \begin{pmatrix} k_{t+1}-k^* \\ c_{t+1}-c^* \end{pmatrix} &= \frac{1}{1+\mu} \begin{pmatrix} \beta^{-1} & -1 \\ R_A^{-1}(c^*)h''(k^*) & -\beta R_A^{-1}(c^*)h''(k^*) + (1+\mu) \end{pmatrix} \begin{pmatrix} k_t-k^* \\ c_t-c^* \end{pmatrix} \\ &= \Phi \begin{pmatrix} k_t-k^* \\ c_t-c^* \end{pmatrix} \end{aligned} \tag{4.16}$$

where $R_A(c^*)$ equals $-U''(c^*)/U'(c^*) > 0$, the absolute risk aversion coefficient evaluated at the steady state.[6] As discussed in section 1.3.2 and 3.3, the dynamics of the system is determined by the eigenvalues of Φ. Denote the characteristic polynomial of Φ by $\mathcal{P}(\lambda)$ and the corresponding eigenvalues by λ_1 and λ_2, then we have

$$\mathcal{P}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \mathrm{tr}(\Phi)\lambda + \det \Phi$$

with

$$\begin{aligned}
\mathrm{tr}\Phi = \lambda_1 + \lambda_2 &= 1 + [\beta(1+\mu)]^{-1} - \beta R_A^{-1}(c^*)h''(k^*)/(1+\mu) > 2 \\
\det \Phi = \lambda_1\lambda_2 &= \frac{1}{\beta(1+\mu)} > 1 \\
\triangle = (\mathrm{tr}\Phi)^2 - 4\det\Phi &= [1 - (\beta(1+\mu))^{-1}]^2 \\
&\quad + \frac{\beta R_A^{-1}(c^*)h''(k^*)}{1+\mu}\left[\frac{\beta R_A^{-1}(c^*)h''(k^*)}{1+\mu} - 2 - \frac{2}{\beta(1+\mu)}\right] > 0 \\
\mathcal{P}(1) = 1 - \mathrm{tr}\Phi + \det\Phi &= \frac{\beta R_A^{-1}(c^*)h''(k^*)}{1+\mu} < 0
\end{aligned}$$

where $\triangle$ denotes the discriminant of the quadratic equation and where we used the fact that $h'' < 0$. The above inequalities have the following implications for the two eigenvalues:

- $\triangle > 0$ implies that the eigenvalues are real and distinct;
- $\mathrm{tr}\Phi > 2$ implies that at least one eigenvalue is greater than 1;
- $\mathcal{P}(1) < 0$ then implies that they lie on opposite sides of 1;
- $\mathcal{P}(0) = \det\Phi > 1$ finally implies that one eigenvalue, say λ_1, is larger than 1 whereas the second eigenvalue, λ_2, lies between 0 and 1.

Because the eigenvalues are distinct, we can diagonalize Φ as $\Phi = Q\Lambda Q^{-1}$ where Λ is a diagonal matrix with λ_1 and λ_2 on its diagonal. We take $\lambda_1 > 1 > \lambda_2 > 0$. The column of the matrix $Q = (q_{ij})$ consist of the eigenvectors of Φ. Thus, the solution can be written as

$$k_t - k^* = c_1 q_{11}\lambda_1^t + c_2 q_{12}\lambda_2^t \tag{4.17}$$
$$c_t - c^* = c_1 q_{21}\lambda_1^t + c_2 q_{22}\lambda_2^t. \tag{4.18}$$

[6] This concept is intimately related to the intertemporal elasticity of substitution in this context. In particular, for the $U(c) = \frac{c^{1-\alpha}-1}{1-\alpha}$ the coefficient of relative risk aversion $R_R = c\,R_A = \alpha$ is just the inverse of the intertemporal elasticity of substitution.

Because $\lambda_1 > 1$ this represents an unstable system. As time evolves the unstable eigenvalue will eventually dominate. In order to avoid this explosive behavior, we set c_1 equal to zero. The second constant c_2 can be determined from the boundary condition associated with the predetermined variable. Suppose the system starts in period zero and we are given a value k_0 for the capital intensity in this period, then according to equation (4.17) $c_2 = \frac{(k_0 - k^*)}{q_{12}}$, provided $q_{12} \neq 0$. Combining all these elements with the equations (4.17) and (4.18) leads to the equation for the saddle path:

$$c_t = c^* + \frac{q_{22}}{q_{12}} (k_t - k^*) \tag{4.19}$$

Next, we show that the saddle path is upward sloping, i.e. that $\frac{q_{22}}{q_{12}} > 0$ as shown by the red line in Figure 4.3. This can be verified by manipulating the defining equations for the eigenvector corresponding to the second eigenvalue λ_2. They are given by $(\phi_{11} - \lambda_2)q_{12} + \phi_{12}q_{22} = 0$ and $\phi_{21}q_{12} + (\phi_{22} - \lambda_2)q_{22} = 0$. Because $(\phi_{11} - \lambda_2) > 0$ and $\phi_{12} < 0$, q_{12} and q_{22} are of the same sign. The same conclusion is reached using the second equation. This argument also shows that $q_{12} \neq 0$. Suppose that $q_{12} = 0$ then q_{22} must also be zero because $\phi_{12} = \frac{-1}{1+\mu} < 0$. This, however, contradicts the assumption that $(q_{12}, q_{22})'$ is an eigenvector. Note that although the eigenvector is not uniquely determined, its direction and thus the slope of saddle path is.

4.2.4 Some Policy Experiments

In the following we discuss two policies. The first one analyzes the introduction of a tax on the return to capital. The second one investigates the effects of an increase in government expenditures.

Taxation of Capital

Suppose that the government levies a proportional tax on the gross return to capital. For simplicity, we assume that the revenues from the tax are just wasted. Therefore only the Euler equation (4.12) is affected. The new Euler equation then becomes

$$U'(c_t) = \beta(1 - \tau)h'(k_{t+1})U'(c_{t+1}) \tag{4.20}$$

where τ is the tax rate with $0 < \tau < 1$. The transition equation for capital is not altered. The new $\Delta c = 0$ schedule then is

$$\Delta c = 0 : \qquad \beta(1 - \tau)h'(k^*) = \beta(1 - \tau)(f'(k^*) + 1 - \delta) = 1.$$

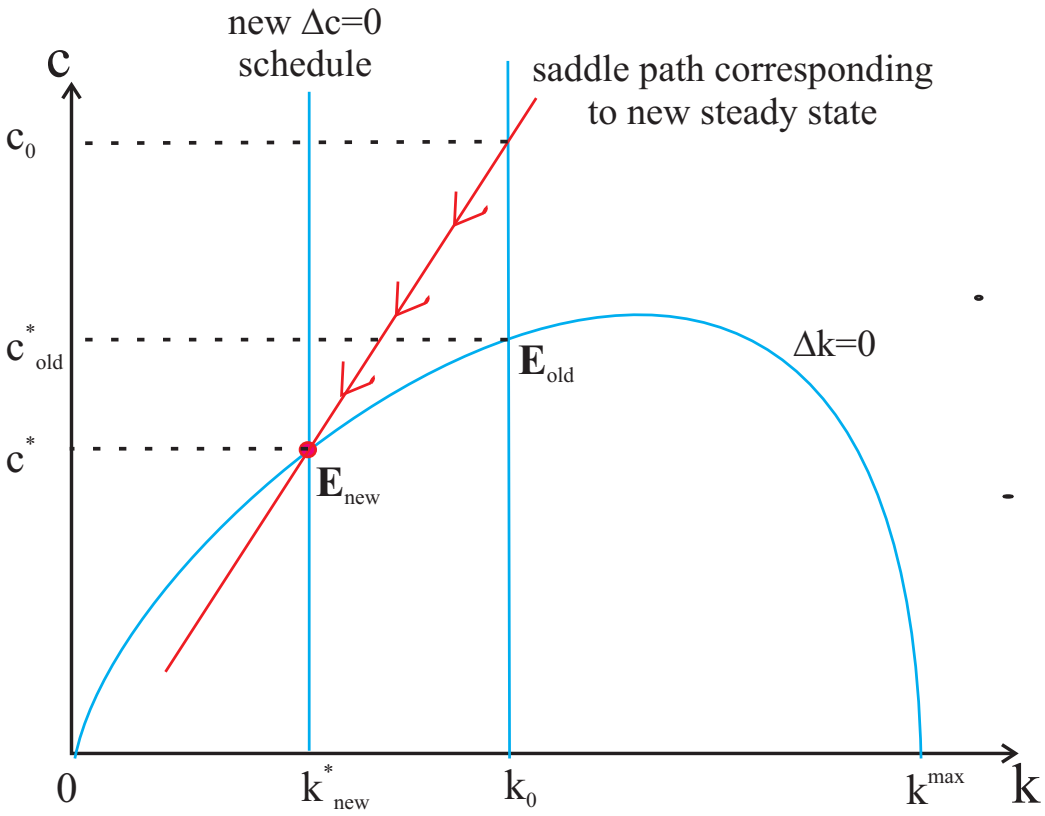


Figure 4.4: Phase diagram of the optimal growth model with distortionary taxation of capital

This implies that an increase in the tax rate τ will lower the steady state capital stock. The tax is thus distortionary.

The evolution of consumption and capital can best be understood by examining the corresponding phase diagram in Figure 4.4. Starting from the initial steady state $\mathbf{E}_{\text{old}}$, the introduction of the tax implies that the new steady state $\mathbf{E}_{\text{new}}$ involves a lower steady state capital stock k^*_{new}. This lower capital stock is achieved by raising consumption. On impact, consumption jumps from c^*_{old} to c_0 such that the point (k_0, c_0) is on the saddle path corresponding to the new steady state. The capital stock remains initially unaffected because it is predetermined. Over time both consumption and capital start to decrease along the new saddle path to reach the new steady state.

Increase in Government Expenditures

Next we consider a permanent increase in government expenditures. These expenditures are deducted from output before the investment/consumption decision by the households takes place. Such a policy will affect only the resource constraint, i.e. the national accounting, but leaves the Euler equation unchanged. Thus, the steady state capital stock remains unchanged by this policy. Given that this policy is not announced in advance, consumption adjusts immediately by falling to its new steady state value. There is no dynamics. Only the distribution of consumption between private and public use is affected.

Denoting government expenditures by g, the national accounting identity (4.11) becomes

$$c_t + g + (1+\mu)k_{t+1} = f(k_t) + (1-\delta)k_t = h(k_t). \tag{4.21}$$

The $\Delta k = 0$ schedule changes accordingly:

$$\Delta k = 0 : \qquad c^* = f(k^*) - (\delta + \mu)k^* - g.$$

Thus, starting initially with no government expenditures, the schedule is shifted down by g as shown in figure 4.5. Consumption reacts immediately and drops from its old steady state value c_0^* to its new one c_1^*.

In contrast to the previously discussed policies which were all unanticipated and *permanent*, we analyze a *transitory* increase in government expenditures which is unanticipated by the time of announcement. More precisely, in period t_0 the government introduces unexpectedly expenditures by an amount $g > 0$, but announces credibly at the same time that it will discontinue this policy in period t_1 and return to $g = 0$. The dynamics of this policy is shown in figure 4.5. On impact at time t_0, consumption will drop from c_0^* to $c_0 > c_1^*$. Thus, the reduction in consumption is smaller than in the permanent case. This puts the system on an unstable path shown in green in figure 4.5. On this path, capital is continuously lowered whereas consumption increases. At time t_1 when the policy is reverted as expected, this unstable path hits the saddle path corresponding to the old steady state. Capital then reaches its lowest value k_1. From then on, the economy moves along the old saddle path back to the old equilibrium $\mathbf{E}_0 : (c_0^*, k^*)$. Capital then starts to grow again. In the long-run the economy moves back to the old steady state.[7]

[7] See Judd (1985) for further details of this type of analysis in a continuous framework.

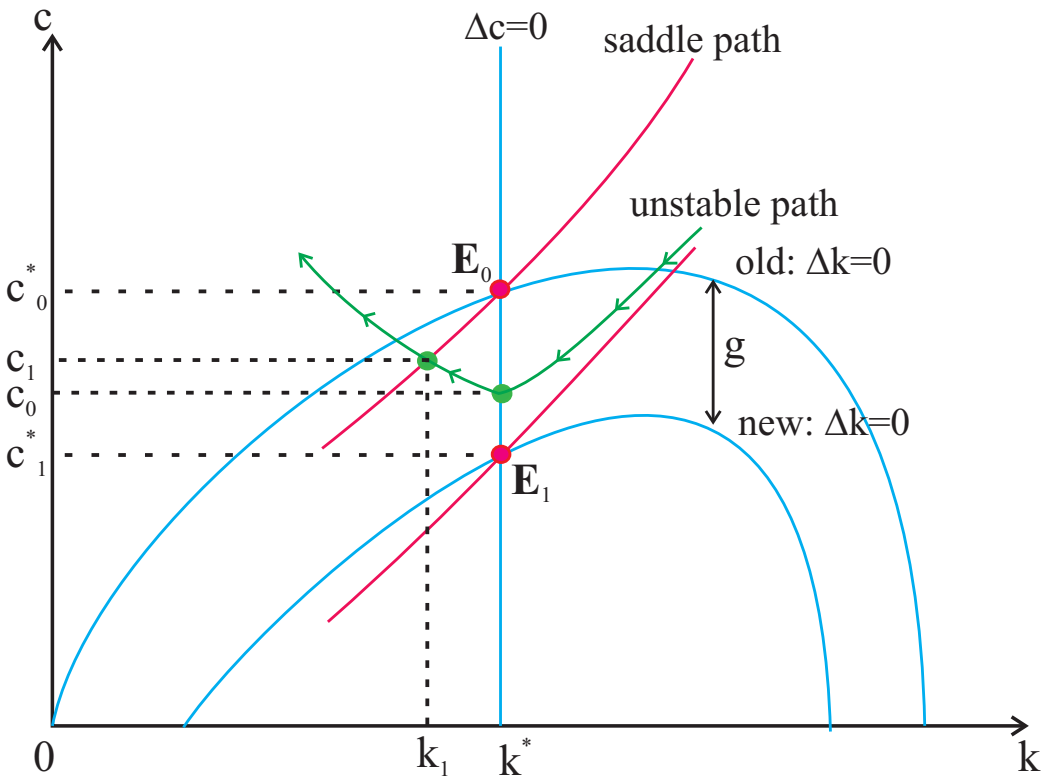


Figure 4.5: Phase diagram of the optimal growth model with government expenditures

4.2.5 An Instructive Special Case

The nonlinear system of difference equations (4.11) and (4.12) has in general no analytical solution so that one has to resort to the linearized version to study its local behavior around the steady state. There is, however, one particular specification which allows for an analytical solution. Set $\mu = 0$ and allow for full depreciation within a period, i.e. $\delta = 1$, furthermore assume that the utility function is logarithmic, i.e $U(c) = \ln c$ and that the production function is Cobb-Douglas, i.e. $f(k) = Ak^\alpha$, $A > 0$ and $0 < \alpha < 1$. With these simplifications, the equation system (4.11) and (4.12) becomes:

$$\begin{aligned} c_t + k_{t+1} &= Ak_t^\alpha \\ \frac{1}{c_t} &= \alpha\beta A k_{t+1}^{\alpha-1} \frac{1}{c_{t+1}} \end{aligned}$$

This system can be rewritten as a nonlinear single equation difference equation in the capital intensity by eliminating c_t and c_{t+1}:

$$\begin{aligned} \frac{1}{Ak_t^\alpha - k_{t+1}} &= \frac{\alpha\beta A k_{t+1}^{\alpha-1}}{Ak_{t+1}^\alpha - k_{t+2}} \\ \frac{1}{(Ak_t^\alpha/k_{t+1}) - 1} &= \frac{\alpha\beta}{1 - (k_{t+2}/Ak_{t+1}^\alpha)}. \end{aligned}$$

This second order equation can be reduced to a first order one by recognizing that $s_{t+1} = \frac{k_{t+2}}{Ak_{t+1}^\alpha}$ and $s_t = \frac{k_{t+1}}{Ak_t^\alpha}$ are the *savings rates* of the economy in periods $t+1$ and t:

$$s_{t+1} = g(s_t) = (1 + \alpha\beta) - \frac{\alpha\beta}{s_t}. \tag{4.22}$$

This nonlinear difference equation is a Riccati type equation which is best analyzed by examining the graph of the function g in Figure (4.6) for $s_t > 0$.[8]

Obviously, there are two steady states $s^* = \alpha\beta$ and $s^* = 1$. The stability of these two fixed points can be characterized by applying Theorem 1.2. Indeed, the derivative of g evaluated at the two fixed points is

$$g'(s^*) = \frac{\alpha\beta}{(s^*)^2} = \begin{cases} \frac{1}{\alpha\beta} > 1, & s^* = \alpha\beta; \\ \alpha\beta < 1, & s^* = 1. \end{cases}$$

Hence, we conclude that $s^* = \alpha\beta$ is an unstable (exploding) fixed point whereas $s^* = 1$ is an exponentially stable fixed point. The special functional form allows for the derivation of further properties. If $s_0 > \frac{\alpha\beta}{1+\alpha\beta}$ the whole

[8] As these type of equations have many interesting properties, they have induced a large literature. An early contribution most relevant for this problem is due to Brand (1955).

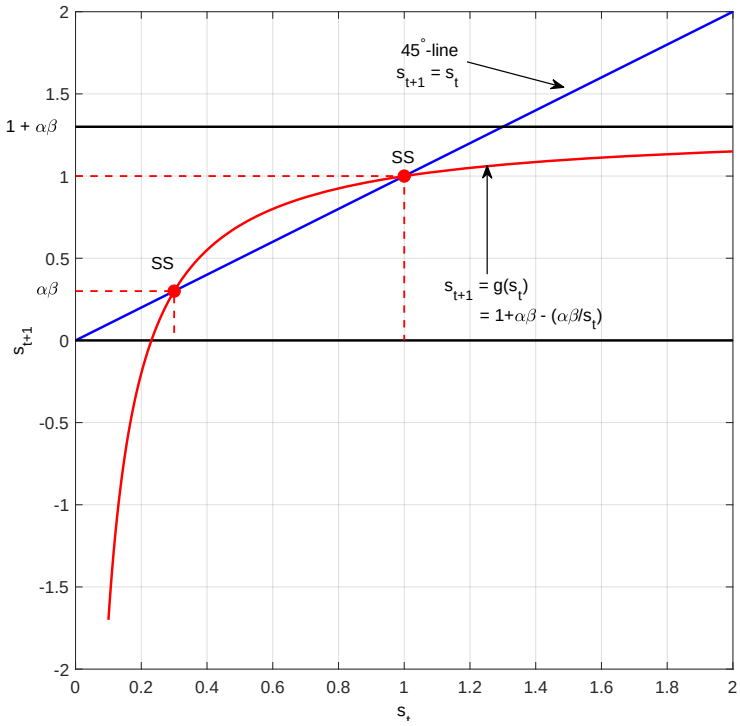


Figure 4.6: Difference equation for the Saving Rate ($\alpha\beta = 0.3$)

orbit will deliver positive savings rates. Moreover, if $s_0 \neq \alpha\beta$, the savings rates converge monotonically to one, i.e. $\lim_{t\to\infty} s_t = \lim_{t\to\infty} \varphi(s_0, t) = 1$. If we allow the economy to borrow at no interest such that s_t can become negative, the fixed point $s^* = 1$ is globally stable, provided $s_0 \neq \alpha\beta$. To see this, enlarge Figure 4.6 to cover also the arm of the hyperbola g for negative savings rates.[9]

We now compare the two fixed points from an economic point of view. $s^* = 1$ means that the agents would consume nothing and save the whole output. Such a situation cannot be optimal. Thus, the only economically sensible fixed point is $s^* = \alpha\beta$. This implies that the savings rate must be set immediately equal to $\alpha\beta$ and remain constant thereafter.

Based on this consideration, the log of k follows a first order linear difference equation:

$$\ln k_{t+1} = \ln(\alpha\beta A) + \alpha \ln(k_t).$$

The steady state for this equation is $\ln(k^*) = \frac{\ln(\alpha\beta A)}{1-\alpha}$. Hence, $\ln k_{t+1} - \ln(k^*) = \alpha(\ln k_t - \ln(k^*))$ and the solution is

$$\ln k_t - \ln(k^*) = \alpha^t(\ln k_0 - \ln(k^*)), \quad \text{for any } k_0 > 0.$$

4.3 A Two-Country Solow Model

In this example we consider two countries A and B. These two countries share the same technology, i.e. they have the same production function $f(k)$ where k is the capital intensity. The savings rate s, $0 < s < 1$, the depreciation rate δ, $0 < \delta \leq 1$, and the population growth rate $\mu > 0$ are also equal in both countries. Hence, in a closed economy context, the capital intensity in both countries follows the fundamental Solow difference equation (2.16):

$$k_{t+1} = \frac{1-\delta}{1+\mu} k_t + \frac{s}{1+\mu} f(k_t) = g(k_t)$$

where f is assumed to the same properties as in Section 2.3.2. The only difference between the two countries is that the capital intensity in period 0 is higher in country A compared t country B. In particular, we assume that

$$k_0^B < k_0^A < k^*$$

where k^* denotes the steady state of the difference equation.

[9] We can allow for $s_t = 0$ by defining $s_{t+1} = \infty$. s_{t+2} then equals $1 + \alpha$ and the subsequent values of s_t decline monotonically to one.

In period 0 both countries agree to open their borders and to engage in trade of newly installed investment goods. Already installed capital and labor are assumed to remain immobile. Savings and therefore investments flow to the country with the higher return. The return on the capital is returned to its owner in the form of consumption goods. Because $k_0^B < k_0^A$, hence $f'(k_0^B) > f'(k_0^A)$, *all* savings accruing in country A will flow as investment to country B. Thus, as long as $k_0^B < k_0^A$, the evolution of the capital intensities in both countries is given by

$$\begin{aligned} k_{t+1}^A &= \frac{1-\delta}{1+\mu} k_t^A, \\ k_{t+1}^B &= \frac{1-\delta}{1+\mu} k_t^B + \frac{s}{1+\mu}\left[f(k_t^A) + f(k_t^B)\right]. \end{aligned}$$

From these equations it is clear that the capital intensity in country A declines monotonically whereas the capital intensity in country B increases monotonically. Because k_{t+1}^B will approach k^* in the limit and because $k_{t+1}^A < k_0^A < k^*$, there exists a time period t^* where both capital intensities are equal.[10] From this period on, both capital intensities will move in parallel and approach the common state k^*. This strong convergence implication can only be partially observed in reality. Some economic obstacles to this convergence hypothesis are discussed in Lucas (1990).

Besides the physical movements of capital it is interesting to account for the income dynamics generated. Let $\tilde{k}_t^B$ denote the capital intensity in country B owned by agents in country A. This entity evolves according to

$$\begin{aligned} \tilde{k}_{t+1}^B &= \frac{s}{1+\mu}\left[f(k_t^A) + f'(k_t^B)\tilde{k}_t^B\right] + \frac{1-\delta}{1+\mu}\tilde{k}_t^B, && t \le t^*, \\ \tilde{k}_{t+1}^B &= \frac{s}{1+\mu} f'(k_t^B)\tilde{k}_t^B + \frac{1-\delta}{1+\mu}\tilde{k}_t^B, && t > t^*. \end{aligned}$$

While the evolution of $\tilde{k}_t^B$ is quite complicated, it is instructive to take a look at the balance of payment for both countries in the two phases.

4.4 The New Keynesian Model

In this section we study a simple version of the New Keynesian macroeconomic model as it has become popular recently. A detailed description of the

[10] In this period savings must eventually be split between the two countries to achieve equality.

model and its microfoundations can be found in Woodford (2003) and Galí (2008). Here we follow the exposition by Galí (2011). The model consists of the following three equations:

$$\begin{aligned} y_t &= y_{t+1} - \frac{1}{\sigma}(i_t - \pi_{t+1}), && \text{(IS-equation)} \\ \pi_t &= \beta\pi_{t+1} + \kappa y_t + u_t, && \text{(forward-looking Phillips-curve)} \\ i_t &= \phi\pi_t, && \text{(Taylor-rule)} \end{aligned}$$

where y_t, π_t, and i_t denote income, inflation and the nominal interest rate, all measured as deviations from the steady state. u_t is an exogenous cost-push shock. Furthermore, we assume that $\sigma > 0$, $\kappa > 0$, and $0 < \beta < 1$. In addition, we take an aggressive central bank, i.e. $\phi > 1$.

This system can be solved for $(y_{t+1}, \pi_{t+1})'$ by inserting the Taylor-rule and Phillips-curve into the IS-equation:

$$\begin{aligned} X_{t+1} = \begin{pmatrix} \pi_{t+1} \\ y_{t+1} \end{pmatrix} &= \frac{1}{\beta} \begin{pmatrix} 1 & -\kappa \\ (\phi\beta - 1)/\sigma & \beta + \kappa/\sigma \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} + \begin{pmatrix} -u_t/\beta \\ u_t/(\sigma\beta) \end{pmatrix} \\ &= \Phi X_t + Z_{t+1} \end{aligned} \tag{4.23}$$

Denote the characteristic polynomial of Φ by $\mathcal{P}(\lambda)$ and the corresponding eigenvalues by λ_1 and λ_2, then we have

$$\mathcal{P}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \mathrm{tr}(\Phi)\lambda + \det\Phi$$

with

$$\begin{aligned} \mathrm{tr}\Phi = \lambda_1 + \lambda_2 &= 1 + \frac{1}{\beta} + \frac{\kappa}{\sigma\beta} > 2 \\ \det\Phi = \lambda_1\lambda_2 &= \frac{1}{\beta} + \frac{\kappa\phi}{\sigma\beta} > 1 \\ \triangle = (\mathrm{tr}\Phi)^2 - 4\det\Phi &= \left(1 - \frac{1}{\beta}\right)^2 + \frac{\kappa}{\sigma\beta}\left(\frac{\kappa}{\sigma\beta} + 2 + \frac{2}{\beta} - 4\phi\right) \\ \mathcal{P}(1) = (1 - \lambda_1)(1 - \lambda_2) &= \frac{\kappa}{\sigma\beta}(\phi - 1) > 0 \quad \text{if } \phi > 1 \end{aligned}$$

where $\triangle$ denotes the discriminant of the quadratic equation. Depending on ϕ, the roots of $\mathcal{P}(\lambda)$ may be complex. We therefore distinguish two cases. First assume that ϕ is high such that $\triangle < 0$. In this case we have two complex conjugate roots. Because $\det\Phi > 1$, they are located outside the unit circle.[11]

[11] Another way to reach this conclusion is by observing that the real part of the roots is $\frac{\mathrm{tr}\Phi}{2} > 1$.

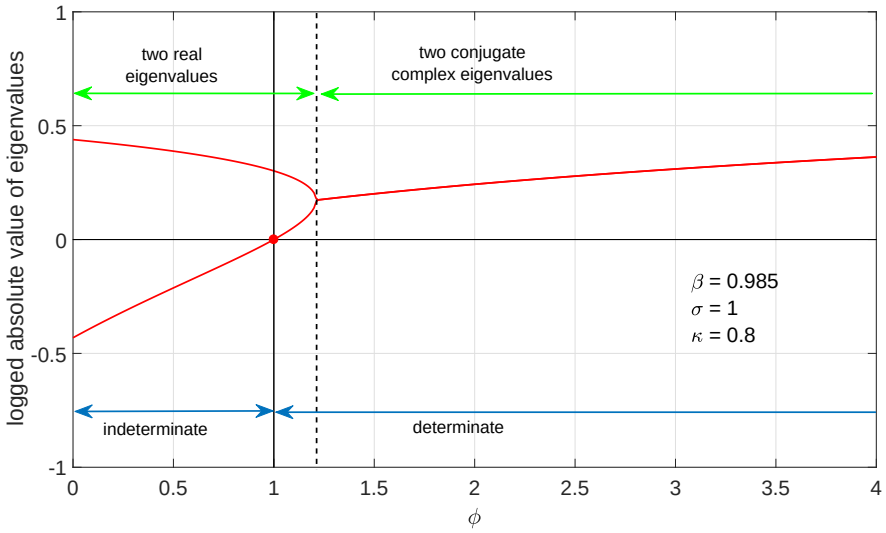


Figure 4.7: Logged absolute values of eigenvalues of the New Keynesian Model

Alternatively assume that ϕ is small enough such that $\triangle > 0$. In this case both eigenvalues are real. Using the assumption $\phi > 1$, $\mathcal{P}(1) > 0$. Thus, both roots are either greater or smaller than one. They cannot be smaller than one because $\text{tr}\Phi > 2$. In both cases we therefore reach the conclusion that the eigenvalues are outside the unit circle. The situation is summarized in Figure 6.2. If ϕ is greater than one, the model is determinate and as both variables are non-predetermined, the boundedness condition (3.27), $Qc = 0$ which is equivalent to $c = 0$, then determines the unique solution:

$$X_t = \sum_{j=1}^{\infty} Q \begin{pmatrix} \lambda_1^{-j} & 0 \\ 0 & \lambda_2^{-j} \end{pmatrix} Q^{-1} \begin{pmatrix} u_{t-1+j}/\beta \\ -u_{t-1+j}/(\sigma\beta) \end{pmatrix}$$

where the columns of Q consist of the eigenvectors corresponding to λ_1 and λ_2.

Suppose now that the central bank fixes the path of the interest rate. The interest rate then becomes an exogenous variable and the system changes to:

$$\begin{aligned} X_{t+1} = \begin{pmatrix} \pi_{t+1} \\ y_{t+1} \end{pmatrix} &= \frac{1}{\beta} \begin{pmatrix} 1 & -\kappa \\ -1/\sigma & \beta + \kappa/\sigma \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} + \begin{pmatrix} -u_t/\beta \\ i_t^*/\sigma + u_t/(\sigma\beta) \end{pmatrix} \\ &= \widetilde{\Phi} X_t + Z_{t+1} \end{aligned} \tag{4.24}$$

where i_t^* is the exogenous path of the interest rate. The trace, determinant,

and discriminant, $\widetilde{\triangle}$ of the characteristic polynomial of $\widetilde{\Phi}$ become:

$$\begin{aligned}
\mathrm{tr}\widetilde{\Phi} &= \lambda_1 + \lambda_2 = 1 + \frac{1}{\beta} + \frac{\kappa}{\sigma\beta} > 2 \\
\det\widetilde{\Phi} &= \lambda_1\lambda_2 = \frac{1}{\beta} > 1 \\
\widetilde{\triangle} &= (\mathrm{tr}\Phi)^2 - 4\det\Phi = \left(1 - \frac{1}{\beta}\right)^2 + \frac{\kappa}{\sigma\beta}\left(\frac{\kappa}{\sigma\beta} + 2 + \frac{2}{\beta}\right) > 0 \\
\mathcal{P}(1) &= (1-\lambda_1)(1-\lambda_2) = -\frac{\kappa}{\sigma\beta} < 0.
\end{aligned}$$

The discriminant is now unambiguously positive so that both eigenvalues are real. Moreover, $\mathcal{P}(1) < 0$ so that one eigenvalue is smaller than one and the other bigger than one. Thus, the boundedness condition does not determine a unique solution. Instead there is a continuum of solutions indexed by c_1 and we are faced with the case of indeterminacy. The implications of this indeterminacy for monetary policy and possible remedies are discussed in Galí (2011).

4.5 The Linear Regulator Problem

Another prototypical case is when the objective function is quadratic and the law of motion linear. Consider the following general setup know as the *Optimal Linear Regulator Problem*:

$$\begin{aligned}
V(x_0) &= -\sum_{t=0}^{\infty} \beta^t \left(x_t' R x_t + u_t' Q u_t\right) \to \max_{u_t}, \qquad 0 < \beta < 1 \qquad (4.25) \\
x_{t+1} &= Ax_t + Bu_t, \qquad x_0 \text{ given},
\end{aligned}$$

where x_t denotes the n-dimensional state vector and u_t the k-vector of controls. R is positive semidefinite symmetric $n \times n$ matrix and Q is a positive definite symmetric $k \times k$ matrix. A and B are $n \times n$, respectively $n \times k$ matrices. Note that the problem has been simplified by allowing no interaction between x_t and u_t. Ljungqvist and Sargent (2018, in particular chapter 5) present many additional theoretical results and economic applications.

One way to solve this optimization problem is to set it up in terms of the *Bellman equation* $V(x)$ which is the value of the objective function starting in state x. Given the additive structure, the value function must satisfy:

$$V(x_t) = \max_{u_t} -\{(x_t' R x_t + u_t' Q u_t) + \beta V(x_{t+1})\} \quad \text{s.t.} \quad x_{t+1} = Ax_t + Bu_t.$$

To solve this functional equation, we guess that $V(x) = -x'Px$ for some positive semidefinite symmetric matrix P. Using this guess and the law of motion the Bellman equation becomes:

$$-x'Px = \max_u -\{x'Rx + u'Qu + \beta(Ax + Bu)'P(Ax + Bu)\}$$

The first order condition of the maximization problem of the right hand side is

$$(Q + \beta B'PB)u = -\beta B'PAx.$$

Thus, we get the feedback rule:

$$u = -Fx = -\beta(Q + \beta B'PB)^{-1}B'PAx.$$

Inserting this rule into the Bellman equation and rearranging terms leads to

$$P = R + \beta A'PA - \beta^2 A'PB(Q + \beta B'PB)^{-1}B'PA.$$

This equation is known as the *algebraic matrix Riccati* equation. It can be solved by iteration:

$$P_{j+1} = R + \beta A'P_jA - \beta^2 A'P_jB(Q + \beta B'P_jB)^{-1}B'P_jA$$

starting from $P_0 = 0$. A sufficient condition for the iteration to converge is that the eigenvalues of A are absolutely strictly smaller than one.

If the optimal rule is inserted into the law of motion, we obtain the *closed-loop solution*:

$$x_{t+1} = (A - BF)x_t$$

This difference equation is stable if the eigenvalues of $A - BF$ are strictly smaller than one in absolute value. There is a large literature investigating the conditions on $R, Q, A,$ and B such a stable closed loop solution obtains. Basically, two conditions must be met. First, A and B must be such that the controller can drive down x_t to zero starting from any initial condition x_0. Second, the R must be such that the controller wants to drive x_t to zero.

As an illustration consider the following simple numerical example with $\beta = 1$, $A = 1$, $B = 1$, $Q = 1$, and $R = 1$. It is instructive to note that this specification allows for the possibility that some plans yield a limit to the infinite sum in (4.25) equal to $-\infty$. Such plans are, however, never optimal as they are dominated by plans with a finite limit. Given the scalar specification, the Riccati equation can be solved analytically:[12]

$$P = 1 + P - \frac{P^2}{1 + P} \quad \Longrightarrow \quad P = (1 + \sqrt{5})/2 \approx 1.618$$

[12]It can also be shown that the Riccati difference equation is stable although $A = 1$ and $\beta = 1$.

The negative solution can be disregarded because we are looking for positive solutions. This implies that

$$u = -Fx = -\frac{P}{1+P}x$$

Inserting this in the law of motion for x_t gives the closed-loop solution:

$$x_{t+1} = \left(1 - \frac{P}{1+P}\right) x_t = \frac{1}{1+P}\, x_t$$

This solution is stable despite $A = 1$ and $\beta = 1$. Finally, $V(x) = P\, x^2 < \infty$.

Chapter 5

Linear Stochastic Expectational Difference Equations

5.1 Introduction and Assumptions

In the following we analyze linear stochastic expectational difference equations in $\{X_t\}$ of the form:

$$\Phi_1 \mathbb{E}_t X_{t+1} = \Phi_0 X_t + Z_t, \qquad t = 0, 1, 2 \ldots, \tag{5.1}$$

where $\{X_t\}$ and $\{Z_t\}$ are real-valued n-dimensional stochastic processes defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The random variables X_t and Z_t are measurable with respect to the σ-algebra $\mathcal{F}_t = \sigma\{(X_s, Z_s) : s \leq t\}$. This makes the sequence $\{\mathcal{F}_t\}$ a filtration adapted to $\{X_t\}$ and $\{Z_t\}$. In economics $\mathcal{F}_t$ is also called the information set. $\mathbb{E}_t$ then denotes the conditional expectation with respect to $\mathcal{F}_t$. As expectations are based on current and past X_t's and Z_t's only and not on extraneous variables, we have eliminated the possibility of sunspot solutions.

Expectational difference equations of the type (5.1) arise typically in the context of rational expectations models. Starting with the seminal paper by Blanchard and Kahn (1980), an extensive literature developed which analyzes the existence and nature of its solutions. The most influential papers, at least for the present exposition, are Gourieroux et al. (1982), Klein (2000), Sims (2001), among others. There is no loss of generality involved by confining the analysis to first order equations as higher order equations can be reduced to first order ones by inflating the dimension of the process (see Binder and Peseran, 1994).

In the following, Φ_1 is not necessarily invertible. Thus, we allow for the possibility that some equations do not involve expectational terms. The

stochastic theory developed below is therefore more general than its deterministic counterpart.

To make the problem tangible, we consider only a certain class of solutions. In particular, we require that the exogenous input process $\{Z_t\}$ is bounded in $\mathfrak{L}^p$, $p > 1$.[1] This means that $\sup_t \|Z_t\|_p < \infty$. This assumption implicitly restricts the solution processes to be bounded as well.

Assumption 5.1. We restrict the class of processes $\{Z_t\}$ to processes bounded in $\mathfrak{L}^p$, $p > 1$. Thus, $\sup_t \mathbb{E}\|Z_t\|_p < \infty$.

Remark 5.1. The class of stationary processes is the prime example of such processes as the expected value and the variance remain constant. In many applications, $\{Z_t\}$ is specified as an ARMA-process.[2]

Throughout the analysis we follow King and Watson (1998) and assume that the linear matrix pencil $\Phi_1 z + \Phi_0$ is regular:[3]

Assumption 5.2. The linear matrix pencil $\Phi_1 z + \Phi_0$ is regular, i.e. there exists at least one $z \in \mathbb{C}$ such that $\det(\Phi_1 z + \Phi_0) \neq 0$.

Remark 5.2. If the matrix pencil $\Phi_1 z + \Phi_0$ would not be regular, there would exist a polynomial vector P in the forward operator $\mathbb{F}X_t = \mathbb{E}_t X_{t+1}$ such that $P(\mathbb{F})Z_t = 0$ for arbitrary processes $\{Z_t\}$. However, this cannot be the case. Note also that the assumption allows either Φ_1 or Φ_0 or both to be singular. In that it generalizes the common assumption Φ_0 and Φ_1 to be invertible.

As was already pointed out by Blanchard and Kahn (1980), the notion of a predetermined variable or process is key for understanding the nature of the solution.[4] Following Klein (p.1412, 2000), we adopt the following definition.

Definition 5.1. A stochastic process $\{K_t\}$ adapted to the filtration $\{\mathcal{F}_t\}$ is a *predetermined process* if

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[1] $\mathfrak{L}^p$ denotes the space of random variables such that $\|X\|_p = \left(\int \|X\|^p \mathrm{d}\mathbf{P}\right)^{1/p} < \infty$ where inside the integral $\|.\|$ is the Euclidian norm in $\mathbb{R}^n$. If $p = 2$, we obtain the square-integrable random variables. In this case $\mathfrak{L}^p$ becomes a Hilbert space.

[2] ARMA-processes are stationary processes which fulfill a stochastic difference equation of the form $Z_t - a_1 Z_{t-1} - \ldots - a_p Z_{t-p} = U_t + b_1 U_{t-1} + \ldots + b_q U_{t-q}$, $a_p b_q \neq 0$, with $\{U_t\}$ being white noise.

[3] See Gantmacher (1959) for an extensive discussion of matrix pencils and the generalized eigenvalue problem of finding $\lambda \in \mathbb{C}$ such $\det(\Phi_1 z + \Phi_0) = 0$.

[4] Sims (2001) provides an alternative approach which does not rely on an a priori division of the variables into predetermined and non-predetermined ones.

(i) The process of expectational errors $\{\eta_t\}$ with $\eta_{t+1} = K_{t+1} - \mathbb{E}_t K_{t+1}$ is an exogenous martingale difference process;

(ii) $K_0 \in \mathcal{F}_0$ is exogenously given.

This definition is more general than the one given in Blanchard and Kahn (1980) who require that $K_{t+1} = \mathbb{E}_t K_{t+1}$.

Finally, note that the superposition principle still applies in this context. Suppose that there exists two solutions $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$ then $\{X_t\} = \{X_t^{(1)} - X_t^{(2)}\}$ satisfies the homogeneous stochastic expectational difference equation

$$\Phi_1 \mathbb{E}_t X_{t+1} = \Phi_0 X_t.$$

Thus, the general solution of equation (5.1) is

$$X_t = X_t^{(g)} + X_t^{(p)}$$

where $X_t^{(g)}$ denotes the general solution to the homogeneous equation and $X_t^{(p)}$ a particular solution to the nonhomogeneous equation.

5.2 The univariate case

Before turning to the details of the multivariate case, we will lay out the main principles by examining a simple univariate example. Consider as a prototype the Cagan model (Cagan, 1956) which we also analyzed in its deterministic form in Section 2.3.3. As in equation (2.28), the current logged price level, here denoted by X_t, is determined by the expectation of its value tomorrow and some exogenous bounded process $\{Z_t\}$ which is a simple transformation of money supply:

$$\mathbb{E}_t X_{t+1} = \phi X_t + Z_t, \qquad \phi \neq 0. \tag{5.2}$$

The superposition principle implies that we can find a solution in two steps. First, find the general solution to the homogeneous equation

$$\mathbb{E}_t X_{t+1} = \phi X_t. \tag{5.3}$$

Then, second, find a particular solution to the nonhomogeneous equation.

Note that the equation (5.2) can be rewritten as a *first order autoregressive* scheme:

$$X_{t+1} = \mathbb{E}_t X_{t+1} + (X_{t+1} - \mathbb{E}_t X_{t+1}) = \phi X_t + Z_t + \eta_{t+1} \tag{5.4}$$

where the expectational errors $\eta_{t+1} = X_{t+1} - \mathbb{E}_t X_{t+1}$ form a martingale difference sequence.[5] Although this process has no serial correlation, it is not necessarily white noise because its variance may change over time.

[5]This is the representation preferred by Sims (2001).

5.2.1 Solution to the homogeneous equation

In order to find the general solution to the homogeneous equation (5.3) note that $\{M_t\}$ defined as $M_t = \phi^{-t} X_t$ is a martingale with respect to $\{\mathcal{F}_t\}$:

$$\mathbb{E}_t M_{t+1} = \mathbb{E}_t \phi^{-t-1} X_{t+1} = \phi^{-t-1} \mathbb{E}_t X_{t+1} = \phi^{-t-1} \phi X_t = M_t.$$

This suggests that the general solution to the homogeneous equation is of the form

$$X_t^{(g)} = \phi^t M_t \tag{5.5}$$

where $\{M_t\}$ is any martingale defined with respect to $\{\mathcal{F}_t\}$. M_t plays the same role as the constant c in the deterministic case. Given Assumption 5.1, we consider only solutions which are bounded in $\mathfrak{L}^p$, $p > 1$ i.e. for which $\sup_t \|X_t^{(g)}\|_p < \infty$. By the Martingale Convergence Theorem (see Hall and Heyde, 1980), there exists a random variable $M \in \mathfrak{L}^p$ such that $M_t = \mathbb{E}(M|\mathcal{F}_t) = \mathbb{E}_t M$. Moreover, $\|M_t - M\|_p$ converges to zero for $t \to \infty$.

As in the deterministic case, we can distinguish several cases depending on the value of ϕ.

$|\phi| > 1$: An implication of the Martingale Convergence Theorem is that $\mathbb{E}\|M_t\|$ converges to $\mathbb{E}\|M\| < \infty$. Thus, for $\{X_t^{(g)}\}$ to remain bounded M_t must be equal to zero for all t and hence the general solution of the homogeneous equation vanishes.

$\phi = 1$: The general solution is $X_t^{(g)} = M_t$ which converges to M.

$\phi = -1$: No convergent solution exists.

$|\phi| < 1$: In this case, the representation (5.4) implies that the solution follows an autoregressive process of order one. This representation admits a causal representation with respect to the expectational errors. This representation is given by $X_t = \phi^t X_0 + \sum_{j=0}^{t-1} \phi^j \eta_{t-j}$ and is bounded. However, if $\{X_t\}$ is not predetermined, there is no starting value X_0 and we are faced with a situation of *indeterminacy* because any martingale difference sequence defined with respect to $\mathcal{F}_t$ would satisfy the difference equation.

5.2.2 Finding a particular solution

A particular solution can be found by iterating the difference equation forward in time and applying the law of iterated expectations. After k iterations

one obtains:

$$\begin{aligned} X_t &= \phi^{-1}\mathbb{E}_t X_{t+1} - \phi^{-1} Z_t \\ &= \phi^{-1}\mathbb{E}_t \left(\phi^{-1}\mathbb{E}_{t+1} X_{t+2} - \phi^{-1} Z_{t+1}\right) - \phi^{-1} Z_t \\ &= \phi^{-2}\mathbb{E}_t X_{t+2} - \phi^{-1} Z_t - \phi^{-2}\mathbb{E}_t Z_{t+1} \\ &= \dots \\ &= \phi^{-k-1}\mathbb{E}_t X_{t+k+1} - \phi^{-1}\sum_{j=0}^{k} \phi^{-j}\mathbb{E}_t Z_{t+j} \end{aligned}$$

As we are looking for solutions which remain bounded, this suggests to take

$$X_t = -\phi^{-1}\sum_{j=0}^{\infty} \phi^{-j}\mathbb{E}_t Z_{t+j} \tag{5.6}$$

as a particular solution if $|\phi| > 1$. As $\{Z_t\}$ is bounded the expression (5.6) qualifies as a candidate for the particular solution when $|\phi| > 1$. Note that $\{X_t\}$ will be bounded (stationary) for *any* bounded (stationary) process $\{Z_t\}$.

If $|\phi| < 1$, the forward iteration may still make sense if $\mathbb{E}_t\|Z_{t+j}\|$ goes to zero quick enough. Take as example the case where $\{Z_t\}$ follows an autoregressive process of order one, i.e. $Z_t = \rho Z_{t-1} + u_t$ with $u_t \sim \mathrm{WN}(0, \sigma^2)$ and $|\rho| < 1$. This specification implies that $\mathbb{E}_t Z_{t+j} = \rho^j Z_t$. Thus, as long as $|\phi^{-1}\rho| < 1$, the forward solution will exist and will be bounded (stationary). However, this will not be true for every $|\rho| < 1$.

In general, if $|\phi| < 1$, we may consider the equivalent representation (5.4) instead. Iterating this equation *backward*, we obtain:

$$X_t = \phi^t X_0 + \sum_{j=0}^{t-1} \phi^j (Z_{t-j} + \eta_{t-j}).$$

This solution, however, makes only sense when $\{X_t\}$ is predetermined so that X_0 is given.

5.2.3 Example: Cagan's model of hyperinflation

Let us illustrate our findings by taking up again Cagan's model of hyperinflation with rational expectations (see section 2.3). This model leads to the following difference equation in the logged price level $\{p_t\}$:

$$\mathbb{E}_t p_{t+1} = \frac{\alpha - 1}{\alpha} p_t + \frac{m_t}{\alpha}, \qquad \alpha < 0, \tag{5.7}$$

where $\{m_t\}$ is now a stochastic process and where the expected price level is replaced by the conditional expectation of the logged price level.

Given that $\alpha < 0$, the coefficient of p_t, $\phi = (\alpha - 1)/\alpha$, is positive and strictly greater than one. Therefore the only bounded (stationary) solution to the homogeneous equation is $\{X_t^{(g)}\} = 0$ and a particular solution can be found by forward iteration. Thus the solution is:

$$p_t = \frac{1}{1-\alpha} \sum_{j=0}^{\infty} \left(\frac{\alpha}{\alpha - 1} \right)^j \mathbb{E}_t m_{t+j}. \tag{5.8}$$

Suppose that the money supply follows an autoregressive process of order one:

$$m_t = a m_{t-1} + \varepsilon_t, \qquad |a| < 1 \text{ and } \varepsilon_t \sim \text{IID}(0, \sigma^2).$$

The conditional expectation $\mathbb{E}_t m_{t+j}$ then equals $a^j m_t$. Inserting this into equation (5.8) leads to

$$p_t = \frac{1}{1-\alpha} \sum_{j=0}^{\infty} \left(\frac{\alpha}{\alpha - 1} \right)^j a^j m_t = \frac{1}{1 - \alpha(1-a)} m_t.$$

This shows that the relation between the price level and money supply depends on the conduct of monetary policy, i.e. it depends on the autoregressive coefficient a. Thus whenever the monetary authority changes its rule, it affects the relation between p_t and m_t. This cross-equation restriction is viewed by Hansen and Sargent (1980) to be the hallmark of rational expectations. It also illustrates that a simple regression of p_t on m_t can not be considered a structural equation, i.e. cannot uncover the true structural coefficients (α in our case), and is therefore subject to the so-called Lucas-critique (see Lucas (1976)).

A similar conclusion is reached if money supply follows a moving average process of order one instead of an autoregressive process of order one:

$$m_t = \varepsilon_t + \theta \varepsilon_{t-1}, \qquad |\theta| < 1 \text{ and } \varepsilon_t \sim \text{WN}(0, \sigma^2).$$

As $|\theta| < 1$, the process for m_t is invertible and we can express ε_t as

$$\varepsilon_t = m_t - \theta m_{t-1} + \theta^2 m_{t-2} - \dots$$

This then leads to the following conclusions:

$$\begin{aligned} m_{t+1} &= \varepsilon_{t+1} + \theta \varepsilon_t = \varepsilon_{t+1} + \theta(m_t - \theta m_{t-1} + \theta^2 m_{t-2} - \dots) \\ \mathbb{E}_t m_{t+1} &= \theta(m_t - \theta m_{t-1} + \theta^2 m_{t-2} - \dots) \end{aligned}$$

and

$$\mathbb{E}_t m_{t+2} = \mathbb{E}_t \varepsilon_{t+2} + \theta \mathbb{E}_t \varepsilon_{t+1} = 0.$$

The particular solution then becomes

$$\begin{aligned} p_t &= \frac{1}{1-\alpha} m_t + \frac{\alpha\theta}{(\alpha-1)(1-\alpha)} (m_t - \theta m_{t-1} + \theta^2 m_{t-2} - \ldots) \\ &= \left(1 + \frac{\alpha\theta}{\alpha-1}\right) \frac{m_t}{1-\alpha} + \frac{\alpha\theta^2}{(1-\alpha)^2} m_{t-1} - \frac{\alpha\theta^3}{(1-\alpha)^2} m_{t-2} + \ldots \end{aligned}$$

Adding the corresponding expression for θp_{t-1} finally gives:

$$p_t + \theta p_{t-1} = \left(1 + \frac{\alpha\theta}{\alpha-1}\right) \frac{m_t}{1-\alpha} + \theta \frac{m_{t-1}}{1-\alpha}$$

This shows that $\{p_t\}$ follows an autoregressive moving average process of order $(1,1)$, i.e. an ARMA(1,1) process, with respect to $\{m_t\}$. The same remarks as before also apply in this case. Note that the AR-polynomial of p_t and the MA-polynomial of m_t coincide. This remains true for general ARMA-specifications for $\{m_t\}$ (Gourieroux et al., 1982).

5.3 The multivariate case

Following the bulk of the literature, we try to decouple the system into a non-explosive (bounded) and an explosive (unbounded) part. Suppose that there are n_k predetermined variables assembled in $\{X_t^{(k)}\}$ then we can partition the vector X_t as

$$X_t = \begin{pmatrix} X_t^{(k)} \\ X_t^{(d)} \end{pmatrix}.$$

The analysis proceeds by first examining the case where Φ_1 is invertible. This is the specification investigated initially by Blanchard and Kahn (1980).

Φ_1 invertible

The invertibility of Φ_1 implies that we can rewrite equation (5.1) as

$$\mathbb{E}_t X_{t+1} = \Phi X_t + \tilde{Z}_t, \qquad t = 0, 1, 2 \ldots$$

where $\Phi = \Phi_1^{-1}\Phi_0$ and $\tilde{Z}_t = \Phi_1^{-1} Z_t$. Let us further assume that Φ is diagonalizable with $\Phi = Q\Lambda Q^{-1}$, Λ diagonal. As in the discussion of the deterministic case in Section 3.4, we partition Λ as

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$$

such that the eigenvalues in Λ_1 are strictly inside the unit circle whereas those in Λ_2 are strictly outside the unit circle. We disregard the case of eigenvalues on the unit circle.

We make the following assumption with respect to the dimension of Λ_1 and Λ_2.

Assumption 5.3. The dimension of Λ_1 is n_k. Thus, there are exactly as many eigenvalues inside the unit circle as there are predetermined variables. Hence, there are as many non-predetermined variables as there are eigenvalues outside the unit circle. This implies that $X_t^{(k)} = X_t^{(1)}$ and $X_t^{(d)} = X_t^{(2)}$.

We will discuss later what happens if this condition is violated. Partitioning Q and X_t accordingly leads to

$$\begin{pmatrix} \mathbb{E}_t X_{t+1}^{(1)} \\ \mathbb{E}_t X_{t+1}^{(2)} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} Q^{(11)} & Q^{(12)} \\ Q^{(21)} & Q^{(22)} \end{pmatrix} \begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix} + \begin{pmatrix} \tilde{Z}_t^{(1)} \\ \tilde{Z}_t^{(2)} \end{pmatrix}$$

where $Q^{-1} = \begin{pmatrix} Q^{(11)} & Q^{(12)} \\ Q^{(21)} & Q^{(22)} \end{pmatrix}$. Multiplying this equation from the left by Q^{-1} leads to the decoupled system

$$\begin{pmatrix} \mathbb{E}_t Y_{t+1}^{(1)} \\ \mathbb{E}_t Y_{t+1}^{(2)} \end{pmatrix} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} Y_t^{(1)} \\ Y_t^{(1)} \end{pmatrix} + \begin{pmatrix} \bar{Z}_t^{(1)} \\ \bar{Z}_t^{(2)} \end{pmatrix}$$

where $Q^{-1}X_t = Y_t$ and $Q^{-1}\tilde{Z}_t = \bar{Z}_t$, i.e.

$$\begin{aligned} Y_t^{(1)} &= Q^{(11)} X_t^{(1)} + Q^{(12)} X_t^{(2)} \\ Y_t^{(2)} &= Q^{(21)} X_t^{(1)} + Q^{(22)} X_t^{(2)} \\ \bar{Z}_t^{(1)} &= Q^{(11)} \tilde{Z}_t^{(1)} + Q^{(12)} \tilde{Z}_t^{(2)} \\ \bar{Z}_t^{(2)} &= Q^{(21)} \tilde{Z}_t^{(1)} + Q^{(22)} \tilde{Z}_t^{(2)}. \end{aligned}$$

Following the logic of the discussion in Section 5.2, the unique bounded solution for $\{Y_t^{(2)}\}$ is

$$\begin{aligned} Y_t^{(2)} &= -\Lambda_2^{-1} \sum_{j=0}^{\infty} \Lambda_2^{-j} \mathbb{E}_t \bar{Z}_{t+j}^{(2)} = -\Lambda_2^{-1} \sum_{j=0}^{\infty} \Lambda_2^{-j} \mathbb{E}_t \left(Q^{(21)} \tilde{Z}_{t+j}^{(1)} + Q^{(22)} \tilde{Z}_{t+j}^{(2)} \right) \\ &= -\Lambda_2^{-1} \sum_{j=0}^{\infty} \Lambda_2^{-j} \begin{pmatrix} Q^{(21)} & Q^{(22)} \end{pmatrix} \mathbb{E}_t \tilde{Z}_{t+j}. \end{aligned} \tag{5.9}$$

Turn next to the first part of the decoupled equation. Note that the predetermined variable $X_t^{(1)}$ satisfies the identity:

$$X_{t+1}^{(1)} - \mathbb{E}_t X_{t+1}^{(1)} = Q_{11}\left(Y_{t+1}^{(1)} - \mathbb{E}_t Y_{t+1}^{(1)}\right) + Q_{12}\left(Y_{t+1}^{(2)} - \mathbb{E}_t Y_{t+1}^{(2)}\right) = \eta_{t+1}.$$

Inserting this equation into the equation for $Y_{t+1}^{(1)}$, we get

$$\begin{aligned} Y_{t+1}^{(1)} &= \mathbb{E}_t Y_{t+1}^{(1)} + \left(Y_{t+1}^{(1)} - \mathbb{E}_t Y_{t+1}^{(1)}\right) \\ &= \Lambda_1 Y_t^{(1)} + \bar{Z}_t{}^{(1)} + Q_{11}^{-1}\eta_{t+1} - Q_{11}^{-1}Q_{12}\left(Y_{t+1}^{(2)} - \mathbb{E}_t Y_{t+1}^{(2)}\right) \\ &= \Lambda_1 Y_t^{(1)} + \bar{Z}_t{}^{(1)} + Q_{11}^{-1}\eta_{t+1} - Q_{11}^{-1}Q_{12}\varepsilon_{t+1} \end{aligned} \tag{5.10}$$

where $\varepsilon_{t+1} = Y_{t+1}^{(2)} - \mathbb{E}_t Y_{t+1}^{(2)}$ is an exogenous martingale difference process. Thus, equation (5.10) is a first order autoregressive scheme with starting value given by

$$Y_0^{(1)} = Q_{11}^{-1}\left(X_0^{(k)} - Q_{12}Y_0^{(2)}\right).$$

Equations (5.10) and (5.9) determine the solution for Y_t. This step in the derivation is only valid if Q_{11} is invertible. Otherwise, we could not determine the initial values of $Y_t^{(1)}$ from those of $X_t^{(1)}$ and there would be a lack of initial values for Y_t.[6] Hence, Assumption 5.3 is not sufficient for the uniqueness of the solution. In addition, we need the following assumption.

Assumption 5.4. Q_{11} is nonsingular.

Finally, the solution for Y_t can be turned back into a solution for X_t by multiplying Y_t by Q.

We can get further insights into the nature of the solution by assuming that $\{\tilde{Z}_t\}$ is a causal autoregressive process of order one:

$$\tilde{Z}_{t+1} = A\tilde{Z}_t + u_{t+1}, \qquad u_t \sim \mathrm{WN}(0, \sigma^2) \text{ and } \|A\| < 1$$

where $\{u_t\}$ is exogenous. This specification implies that $\mathbb{E}_t\tilde{Z}_{t+j} = A^j\tilde{Z}_t$, $j = 1, 2, \ldots$ Inserting this into equation (5.9) we find that

$$Y_t^{(2)} = -\Lambda_2^{-1}\sum_{j=0}^{\infty}\Lambda_2^{-j}\begin{pmatrix} Q^{(21)} & Q^{(22)}\end{pmatrix}A^j\tilde{Z}_t = M\tilde{Z}_t$$

where $M = -\Lambda_2^{-1}\sum_{j=0}^{\infty}\Lambda_2^{-j}\begin{pmatrix} Q^{(21)} & Q^{(22)}\end{pmatrix}A^j$. The solution to $Y_t^{(1)}$ then can be written as

$$Y_{t+1}^{(1)} = \Lambda_1 Y_t^{(1)} + \bar{Z}_t{}^{(1)} + Q_{11}^{-1}\eta_{t+1} - Q_{11}^{-1}Q_{12}Mu_{t+1}.$$

[6] See Klein (2000, section 5.3.1) and King and Watson (2002) for details and examples.

Finally, the initial condition can be computed as

$$Y_0^{(1)} = Q_{11}^{-1}\left(X_0^{(k)} - Q_{12}M\tilde{Z}_0\right).$$

Before turning to the general case several remarks are in order.

Remark 5.3. The above derivation remains valid even if the matrix Φ is not diagonalizable. In this case, we will have to work with the Jordan canonical form instead (see Section 3.2.2).

Remark 5.4. The derivation excluded the possibility of roots on the unit circle.

Φ_1 singular

In many practical applications, Φ_1 is not invertible so that the procedure just outlined is not immediately applicable. This, for example, is the case when a particular equation contains no expectations at all which translates into a corresponding row of zeros in Φ_1. One way to deal with this problem is to take a generalized inverse of Φ_1 and proceed as explained above.

The most appropriate type of generalized inverse in the context of difference equations is the *Drazin-inverse* (see Campbell and Meyer, 1979, for a comprehensive exposition). This generalized inverse can be obtained for any $n \times n$ matrix A in the following manner. Denote by Ind$A$ the smallest nonnegative integer k such that rankA^k = rankA^{k+1}. This number is called the *index* of A. Then the following Theorem holds (see Theorem 7.2.1 in Campbell and Meyer, 1979).

Theorem 5.1. Let A be an $n \times n$ matrix with $\text{Ind}(A) = k > 0$, then there exists a nonsingular matrix P such that

$$A = P\begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix}P^{-1}$$

where C is nonsingular and N is nilpotent of index k (i.e. $N^k = 0$). The Drazin-inverse A^D is then given by

$$A^D = P\begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix}P^{-1}.$$

With this Theorem in mind, we can now decouple the system in two parts. The first part will be similar to the case when Φ_1 is invertible. The second one will correspond to the singular part and will in some sense solve

out the expectations. Multiply for this purpose equation (5.1) from the left by $(z\Phi_1 + \Phi_0)^{-1}$ where assumption 5.2 guarantees that the inverse exists for some number z. Thus, we get

$$\begin{aligned}(z\Phi_1 + \Phi_0)^{-1}\Phi_1\mathbb{E}_t X_{t+1} &= (z\Phi_1 + \Phi_0)^{-1}\Phi_0 X_t \\ \widehat{\Phi}_1\mathbb{E}_t X_{t+1} &= (z\Phi_1 + \Phi_0)^{-1}(z\Phi_1 + \Phi_0 - z\Phi_1)X_t \\ \widehat{\Phi}_1\mathbb{E}_t X_{t+1} &= (I_n - z\widehat{\Phi}_1)X_t\end{aligned}$$

where $\widehat{\Phi}_1 = (z\Phi_1 + \Phi_0)^{-1}\Phi_1$. The application of Theorem 5.1 to $\widehat{\Phi}_1$ then leads to the decoupled system

$$\begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix} E_t\widetilde{X}_{t+1} = \left(I_n - z\begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix}\right)\widetilde{X}_t$$

where $\widetilde{X}_t$ denotes $P^{-1}X_t$. This leads to the following two equations:

$$\begin{aligned}C\mathbb{E}_t\widetilde{X}^{(1)}_{t+1} &= (I_{n_1} - zC)\widetilde{X}^{(1)}_t \\ N\mathbb{E}_t\widetilde{X}^{(2)}_{t+1} &= (I_{n_2} - zN)\widetilde{X}^{(2)}_t\end{aligned}$$

where $\widetilde{X}_t$ has been partitioned appropriately. As C is invertible, the first difference equation can be treated as outlined above. By shifting the time index, the second equation can be written as

$$N\mathbb{E}_t\widetilde{X}^{(2)}_{t+k} = (I_{n_2} - zN)\widetilde{X}^{(2)}_{t+k-1}.$$

Applying the law of iterated expectations and multiplying the equation from the left by N^{k-1} gives

$$\begin{aligned}0 = N^k\mathbb{E}_t\widetilde{X}^{(2)}_{t+k} &= (I_{n_2} - zN)^2N^{k-2}\mathbb{E}_t\widetilde{X}^{(2)}_{t+k-2} \\ &= (I_{n_2} - zN)^3N^{k-3}\mathbb{E}_t\widetilde{X}^{(2)}_{t+k-3} \\ &= \dots \\ &= (I_{n_2} - zN)^k\mathbb{E}_t\widetilde{X}^{(2)}_t \\ &= (I_{n_2} - zN)^k\widetilde{X}^{(2)}_t.\end{aligned}$$

Because $(I_{n_2} - zN)^k$ is invertible, the only solution to the above equation is $\widetilde{X}^{(2)}_t = 0$.

Chapter 6

Systems with Time–Varying Coefficients

Up to now we have assumed that the coefficients of the homogeneous part (the coefficients in Φ) are constant. This presupposition, although convenient and appropriate in many instances, is a very tenuous position. Indeed, there are several convincing reasons to believe in time-varying coefficients instead. First, time-varying coefficient models arise naturally from the linearization of nonlinear models along solution paths as shown in the next section (see also Elaydi, 2005, p.219–220). Second, the relationships describing the economy undergo structural changes giving rise to drifting coefficients as emphasized by Lucas' critique (Lucas, 1976). Sargent (1999), for example, provides an interpretation in terms of self-confirming equilibria and learning. Third, policies and policy rules are subject to change. Cogley and Sargent (2005), Primiceri (2005), or Chen et al. (2015) provide empirical evidence with to U.S. monetary policy.

While the widening in scope is a deserving undertaking, it requires the introduction of new concepts which also increase the mathematical level of analysis. As the standard eigenvalue/eigenvector analysis exposed previously does not deliver purposeful results, we must introduce Lyapunov exponents, respectively Lyapunov spaces, as alternatives. As it turns out the Lyapunov exponents and Lyapunov spaces provide just the right tools and can be considered as perfect substitutes for the eigenvalues and eigenspaces. The exposition heavily relies on the monographs Argyris et al. (2015), Arnold (2003), Colonius and Kliemann (2014), and Viana (2014). We start the presentation by motivating these concepts and applying them to the constant coefficient case. Thereby the relation between eigenvalues/eigenspaces and Lyapunov exponents / Lyapunov spaces is established. We will then consider periodically moving coefficients which will show the limits of the

eigenvalues/eigenspaces analysis. We then turn to the case of deterministically time-varying coefficients. Finally, we will allow the coefficients to vary stochastically.

6.1 Lyapunov Exponents and Lyapunov Spaces

Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable transformation. This transformation gives rise to a homogeneous difference equation:

$$x_{t+1} = f(x_t).$$

Given a starting value x in period 0, this difference equation generates a solution (orbit) $x_t = \varphi(t, x) = f^t(x)$ which we take as a reference. Consider a new solution $\bar{x}_t = \varphi(t, x + \Delta x)$ obtained from perturbing the initial condition by a small change Δx. Hence, we have $\bar{x}_{t+1} = x_{t+1} + \Delta x_{t+1} = f(x_t + \Delta x_t)$. Taking a first order Taylor expansion (linearization) around x_t, we get

$$\bar{x}_{t+1} = f(x_t) + f'(x_t)\Delta x_t = x_{t+1} + f'(x_t)\Delta x_t.$$

Hence, for the perturbation we get

$$\Delta x_{t+1} = f'(x_t)\Delta x_t. \tag{6.1}$$

This has the form of a first order homogenous linear difference equation. However, its coefficient $f'(x_t)$ depends on t and is *not constant* over time. Iterating this equation $t + 1$-times,

$$\begin{aligned} \Delta x_{t+1} &= f'(x_t)f'(x_{t-1}) \dots f'(x_0)\Delta x \\ &= \prod_{\tau=0}^{t} f'(x_\tau)\Delta x \end{aligned}$$

where we have identified Δx with Δx_0. The Lyapunov exponent λ is then defined as the mean exponential divergence or converge of neighbouring solutions $|\Delta x_t| \approx \mathrm{e}^{t\lambda}|\Delta x|$ for $t \to \infty$ and $|\Delta x| \to 0$. Hence, the Lyapunov exponent for univariate difference equations is obtained as

$$\lambda(x) = \lim_{t\to\infty} \lim_{|\Delta x|\to 0} \frac{1}{t} \log \frac{\Delta x_t}{\Delta x} = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \log |f'(x_\tau)|.$$

The dependence on x arises because we are considering perturbations from a solution starting in x. Thus, we allow for the possibility that solution with

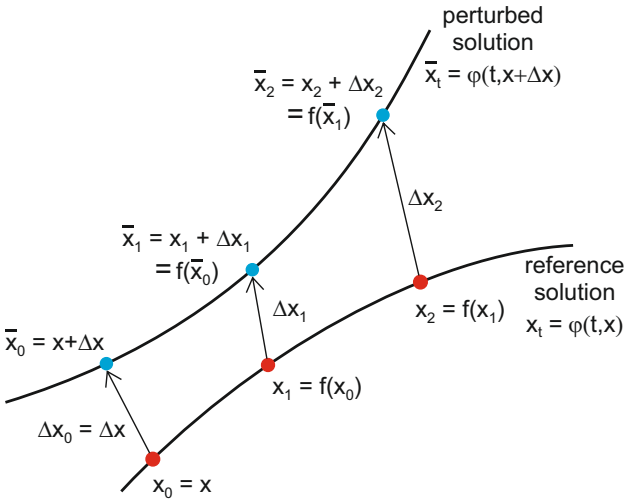

Figure 6.1: Definition of Lyapunov Exponents

different starting values will have different Lyapunov exponents. In order to ensure the existence of such a number, the limit is replaced by a *limes superior*. A special reference solution is given by a solution starting in a steady state. Hence, the Lyapunov exponent provides information about the stability of a steady state. In particular, if the Lyapunov exponent is negative, the influence of the transient dynamics is eliminated and the perturbation of the initial value vanishes in the long run. Figure 6.1 gives an intuitive interpretation of the Lyapunov exponent.

The definition of Lyapunov exponents can be extended to multivariate transformations in a straightforward manner. If the system is generated by a continuously differentiable transformation $f : \mathbb{R}^d \to \mathbb{R}^d$, the first order linear homogenous difference equation corresponding to equation (6.1) is given by

$$\Delta x_{t+1} = \mathrm{D}f(x_t)\Delta x_t$$

where $\mathrm{D}f(x_t)$ denotes the Jacobian matrix of f evaluated at x_t. Differentiating $x_t = f^t(x)$ by applying the chain rule leads to

$$\Delta x_t = \mathrm{D}f^t(x)\Delta x = \underbrace{\mathrm{D}f(f^{t-1}(x_{t-1}))\ldots\mathrm{D}(f(f(x))\mathrm{D}f(x)}_{t\text{-fold product of time-varying matrices}}\Delta x$$

where $\mathrm{D}f^t(x)$ gives rise to a t-fold matrix product of time-varying matrices. This demonstrates how time-varying coefficient matrices arise naturally

through linearization. As in the univariate case, we define the Lyapunov exponent $\lambda(x)$ as

$$\lambda(x) = \limsup_{t\to\infty} \frac{1}{t} \log \frac{\|\Delta x_t\|}{\|\Delta x\|} = \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi(t,x)\|$$

where $\varphi(t,x) = \mathrm{D}f(f^{t-1}(x_{t-1})) \dots \mathrm{D}(f(f(x))\mathrm{D}f(x)x$. Because the state space is finite dimensional, this definition is independent of the norm.

Lemma 6.1. The Lyapunov exponent is independent of the norm actually chosen.

Proof. Because the state space is finite dimensional, all norms are equivalent. Let $\|.\|_a$ and $\|.\|_b$ two norms on $\mathbf{X} = \mathbb{R}^d$. Then there exists positive constants c_1 and c_2 such that

$$\frac{1}{t} \log(c_1 \|\varphi(t,x)\|_b) \leq \frac{1}{t} \log(\|\varphi(t,x)\|_a) \leq \frac{1}{t} \log(c_2 \|\varphi(t,x)\|_b)$$

or equivalently

$$\frac{1}{t}(\log c_1 + \log \|\varphi(t,x)\|_b) \leq \frac{1}{t} \log(\|\varphi(t,x)\|_a) \leq \frac{1}{t}(\log c_2 + \log \|\varphi(t,x)\|_b).$$

Taking the limit for $t \to \infty$,

$$\limsup \frac{1}{t} \log \|\varphi(t,x)\|_b \leq \limsup \frac{1}{t} \log(\|\varphi(t,x)\|_a) \leq \limsup \frac{1}{t} \log \|\varphi(t,x)\|_b.$$

Hence, the Lyapunov exponents with respect to the two norms are equal. □

From this definition we immediately deduce the following two characterising properties.[1]

Proposition 6.1 (Properties of Lyapunov Exponents)**.** The Lyapunov exponents satisfy:

(i) For any real number $c \neq 0$, $\lambda(cx) = \lambda(x)$.

(ii) For any two vectors x and y, $\lambda(x+y) \leq \max\{\lambda(x), \lambda(y)\}$.

Proof. The definition of the Lyapunov exponent implies:

$$\lambda(cx) = \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi(t,cx)\| = \limsup_{t\to\infty} \frac{1}{t}(\log|c| + \log \|\varphi(t,x)\|) = \lambda(x).$$

[1] See Arnold (2003, section 3.2.1) and the literature cited there for more details.

This establishes the first characteristic. Assume that $\max\{\lambda(x), \lambda(y)\} = \lambda(x)$, then there exists a time τ such that for all $t > \tau$,

$$\frac{1}{t}\log(\|x+y\|) \leq \frac{1}{t}\log(\|x\|+\|y\|) = \frac{1}{t}\log\|x\| + \frac{1}{t}\log\left(1+\frac{\|y\|}{\|x\|}\right).$$

This establishes the second characteristic as t goes to ∞. □

The two characteristics imply that the sets $\mathbb{V}_c = \{x \mid \lambda(x) \leq c\}$, $c \in \mathbb{R}$, are linear subspaces of $\mathbb{R}^d$. Indeed, for all $x, y \in \mathbb{V}_c$ and $\alpha \in \mathbb{R}$, $x + y \in \mathbb{V}_c$ and $\alpha x \in \mathbb{V}_c$. Hence, $\mathbb{V}_c$ is closed under vector addition and multiplication by a scalar. To determine the number of different Lyapunov exponents, we establish the lemma.

Lemma 6.2. Vectors with different Lyapunov exponents are linearly independent.

Proof. Take two vectors $x, y \in \mathbb{R}^d \setminus 0$ with $\lambda(x) \neq \lambda(y)$ and suppose that they are linearly dependent. Thus, there exists $\alpha, \beta \in \mathbb{R}$, not equal to zero, such that $\alpha x + \beta y = 0$. Hence, $y = -(\alpha/\beta)x$. The first characteristic of Lyapunov exponents then implies x and y must have the same Lyapunov exponent. This a contradiction, hence x and y are linearly independent. □

Because there can be at most d linearly independent vectors in $\mathbb{R}^d$, the maximal number of Lyapunov exponents is d. Suppose that there are $1 \leq \ell \leq d$ different Lyapunov exponents and order them as

$$\lambda_1 > \cdots > \lambda_\ell.$$

Then define the ℓ linear subspaces $\mathbb{V}_i = \{x \mid \lambda(x) \leq \lambda_i\}$, $i = 1, \ldots, \ell$. They form a *flag* of subspaces (*filtration*) of $\mathbb{R}^d$:

$$\{0\} \subset \mathbb{V}_\ell \subset \ldots \subset \mathbb{V}_1 = \mathbb{R}^d$$

where the inclusions are proper. Moreover,

$$\lambda(x) = \lambda_i \Longleftrightarrow x \in \mathbb{V}_i \setminus \mathbb{V}_{i+1}, \quad i = 1, \ldots, \ell,$$

where $\mathbb{V}_{\ell+1}$ is identified with $\{0\}$.

6.2 Fundamental Matrix and Green's Matrix

The following sections deal with the case where f is an affine transformation:

$$x_{t+1} = A_t x_t + b_t, \qquad t \in \mathbb{Z},$$

where the matrices $A_t \in \mathbb{GL}(d)$ are allowed to vary either deterministically or randomly. In order to analyze such systems it is useful to define the matrix product $\Phi(t)$:

$$\Phi(t) = \begin{cases} A_{t-1} \dots A_1 A_0, & t = 1, 2, \dots; \\ I_d, & t = 0; \\ A_t^{-1} \dots A_{-1}^{-1}, & t = -1, -2, \dots \end{cases} \tag{6.2}$$

Because $\Phi(t)$ satisfies the matrix linear difference equation $\Phi(t+1) = A_t \Phi(t)$, $\Phi(t)$ is a *fundamental matrix* for the linear time-varying system $x_{t+1} = A_t x_t$. It is even a *principal fundamental matrix* because $\Phi(0) = I_d$.[2] More generally, define for any fundamental matrix $\Phi(t)$ the matrix

$$\Phi(t, s) = \Phi(t)\Phi^{-1}(s), \qquad t, s \in \mathbb{Z}.$$

The properties of $\Phi(t, s)$ are summarized in the following lemma.

Lemma 6.3. For all $s, t \in \mathbb{Z}$

(i) $\Phi(t+1, s) = A_t \Phi(t, s)$. Hence, $\Phi(t, s)$ is a solution of the matrix difference equation, therefore a fundamental matrix;

(ii) $\Phi(t, 0)$ is a principal fundamental matrix;

(iii) $\Phi^{-1}(t, s) = \Phi(s, t)$;

(iv)
$$\Phi(t, s) = \begin{cases} \prod_{j=1}^{t-s} A_{t-j} = A_{t-1} A_{t-2} \dots A_s, & t > s; \\ I_d, & t = s; \\ \prod_{j=t}^{s-1} A_j = A_t^{-1} A_{t+1}^{-1} \dots A_{s-1}^{-1}, & t < s. \end{cases}$$

In accordance with Section 3.2.1, when A_t is constant and equal to A, $\Phi(t, s) = A^{t-s}$.

As in Section 3.2.1, *Green's matrix* is defined as

$$\Gamma(t, s) = \Phi(t, r)\Phi^{-1}(s, r), \qquad r, s, t \in \mathbb{Z}.$$

[2] The concept of a fundamental matrix was already introduced in Section 3.2.1 for the constant coefficient case. With time-varying coefficients the concept is becoming more relevant.

Lemma 6.4. For all $r, s, t \in \mathbb{Z}$ Green's matrix satisfies:

(i) $\Gamma(t,t) = I_d$.

(ii) $\Gamma^{-1}(t,s) = \Gamma(s,t)$;

(iii) $\Gamma(t+1,s) = A_t\Gamma(t,s)$ and $\Gamma(t,s+1) = \Gamma(t,s)A_t^{-1}$;

(iv) $\Gamma(t,s) = \Gamma(t,r)\Gamma(r,s)$;

(v)
$$\Gamma(t,s) = \begin{cases} \prod_{j=1}^{t-s} A_{t-j} = A_{t-1}A_{t-2}\dots A_s, & t > s; \\ I_d, & t = s; \\ \prod_{j=t}^{s-1} A_j = A_t^{-1}A_{t+1}^{-1}\dots A_{s-1}^{-1}, & t < s. \end{cases}$$

In accordance with Section 3.2.1, when A_t is constant and equal to A, $\Gamma(t,s) = A^{t-s}$.

6.3 Constant Coefficients

In order to familiarize with the notion of Lyapunov exponents, we revisit the case of linear constant coefficient first order difference equations generated by $f(x) = Ax$:

$$x_{t+1} = Ax_t, \qquad A \in \mathbb{GL}(d). \tag{6.3}$$

For this system $x^* = 0$ is a steady state and solutions are given by $x_t = \varphi(t,x) = A^t x$. Taking the zero solution as the reference solution, the Lyapunov exponent is

$$\lambda(x) = \limsup \frac{1}{t} \log \|\varphi(t,x)\| = \limsup \frac{1}{t} \log \|A^t x\|.$$

Before establishing the relationship between Lyapunov exponents and eigenvalues, we analyze how a similarity transformation of A affects the Lyapunov exponents.

Lemma 6.5. Let $A, B \in \mathbb{GL}(d)$ be related by a similarity transformation, i.e. there exists $Q \in \mathbb{GL}(d)$ such that $B = Q^{-1}AQ$. Then the Lyapunov exponents of the solution $\varphi_A(t,x)$ of $x_{t+1} = Ax$ and $\varphi_B(t,x)$ of $x_{t+1} = Bx$ are related by

$$\lambda_B(x) = \lambda_A(Qx).$$

Proof. Note that

$$\varphi_A(t, Qx) = A^t Qx = QQ^{-1}A^t Qx = QB^t x = Q\varphi_B(t, x).$$

Hence, $\varphi_B(t, x) = Q^{-1}\varphi_A(t, Qx)$. Using this relation, we find

$$\begin{aligned}\lambda_B(x) &= \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi_B(t, x)\| = \limsup_{t\to\infty} \frac{1}{t} \log \|Q^{-1}\varphi_A(t, Qx)\| \\ &\leq \limsup_{t\to\infty} \frac{1}{t} \log \|Q^{-1}\| + \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi_A(t, Qx)\| = \lambda_A(Qx).\end{aligned}$$

Similarly,

$$\begin{aligned}\lambda_A(Qx) &= \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi_A(t, Qx)\| = \limsup_{t\to\infty} \frac{1}{t} \log \|QQ^{-1}\varphi_A(t, Qx)\| \\ &\leq \limsup_{t\to\infty} \frac{1}{t} \log \|Q\| + \limsup_{t\to\infty} \frac{1}{t} \log \|Q^{-1}\varphi_A(t, Qx)\| = \lambda_B(x).\end{aligned}$$

Combining the two inequalities leads the assertion. □

We are now in a position to state the main theorem of this section.

Theorem 6.1 (Lyapunov Exponents and Eigenvalues)**.** Consider the linear difference equation (6.3). Then the state space $\mathbf{X} = \mathbb{R}^d$ can be decomposed into subspace $\mathbb{L}_i$, $i = 1, \ldots, \ell \leq d$, called Lyapunov spaces, such that

$$\mathbb{R}^d = \mathbb{L}_1 \oplus \ldots \oplus \mathbb{L}_\ell.$$

The corresponding Lyapunov exponents $\lambda(x)$, $x \in \mathbb{R}^d$, are given as the logarithms λ_i of the moduli of the eigenvalues of A. For a solution $\varphi(t, x)$, $x \neq 0$, one has

$$\lambda(x) = \lim_{t\to\pm\infty} \frac{1}{t} \log \|\varphi(t, x)\| = \lambda_i \quad \text{if and only if} \quad x \in \mathbb{L}_i.$$

Proof. Colonius and Kliemann (2014, section 1.5) □

Several remarks are in order.

Remark 6.1. (i) The limes superior is actually a normal limit.

(ii) As $A \in \mathbb{GL}(d)$, A is invertible. Hence, zero is not an eigenvalue. Therefore all Lyapunov exponets are finite, i.e. $\lambda_\ell > -\infty$.

(iii) It is important to take the two-sided limit, $t \to \infty$ and $t \to -\infty$. Any starting value $x = x_1 + x_2$ with $x_i \in \mathbb{L}_i$, $\lambda_1 > \lambda_2$, and $x_1 \neq 0$, has Lyapunov exponent $\lambda(x) = \lambda_1$.

(iv) For the univariate case $x_{t+1} = ax_t$ he have

$$\lambda(x) = \lim_{t\to\pm\infty} \frac{1}{t} \log |a^t x| = \lim_{t\to\pm\infty} \frac{1}{t} \log(|a|^t) + \lim_{t\to\pm\infty} \frac{1}{t} \log |x| = \log |a|.$$

(v) We may define a time-reversed difference equation as follows. Define $y_t = x_{-t}$. Then, $y_{t+1} = x_{-t-1} = A^{-1}x_{-t} = A^{-1}y_t$ for all $t \in \mathbb{Z}$. Thus, $x_{t+1} = A^{-1}x_t$, $t \in \mathbb{Z}$, defines the time-reversed equation. Because the eigenvalues of A^{-1} are by the inverses of the eigenvalues of A, the Lyapunov exponents of the time-reversed equation are $-\lambda_i$, but the Lyapunov spaces coincide.

Theorem 6.1 has immediate consequences for characterizing the stability of the zero fixed point of the linear system 6.3. For this purpose it is useful to make the following definitions.

Definition 6.1 (Stable, Center, Unstable Subspaces)**.** The stable, center, and unstable subspaces associated with $A \in \mathbb{GL}(d)$, respectively with the linear difference equation (6.3), are defined as

$$\mathbb{L}^s = \bigoplus_{\lambda_j<0} \mathbb{L}_j, \quad , \mathbb{L}^0, \quad , \mathbb{L}^u = \bigoplus_{\lambda_j>0} \mathbb{L}_j$$

where $\mathbb{L}^0$ is the Lyapunov space corresponding to Lyapunov exponents equal to zero.

This definitions give rise to the following theorem.

Theorem 6.2. The zero fixed point of system (6.3) is asymptotically stable, hence exponentially stable, if and only if all Lyapunov exponents are negative, or equivalently if $\mathbb{L}^s = \mathbb{R}^d$. Moreover, the zero fixed point is stable if and only all Lyapunov exponents are nonpositive and the eigenvalues with modulus equal to one are semisimple.

Proof. This an immediate consequence of Theorem 1.3. □

Thus, we see that Lyapunov exponents do characterize asymptotic (exponential) stability, but not necessarily stability.

The solution of the affine difference equation $x_{t+1} = Ax_t + b_t$ is then found as shown in Section (3.4).

6.4 Periodically Time-Varying Coefficients

In this Section we broaden the analysis and consider difference equations of the following type:

$$x_{t+1} = \psi_t(x_t), \qquad t \in \mathbb{Z},$$

where $\psi_t : \mathbb{R}^d \to \mathbb{R}^d$ is an affine map which varies deterministically with time. This time variation takes place independent from the state of the system. More specifically,

$$\psi_t(x_t) = A_t x_t + b_t, \qquad t \in \mathbb{Z} \text{ and } A_t \in \mathbb{GL}(d). \tag{6.4}$$

The reader can convince himself that the superposition principle also holds in this case. We therefore start with the analysis of the linear nonautonomous case

$$x_{t+1} = A_t x_t. \tag{6.5}$$

with periodically varying coefficients. In particular,

$$A_{t+p} = A_t, \quad t \in \mathbb{Z}, \text{ for some } p \in \mathbb{N}.$$

Denote by $\varphi(t, \nu, x)$ the solution of $x_{t+1} = A_t x_t$ with initial condition x and $A_t = A_{\theta(t+\nu)}$, $\nu = 0, 1, \ldots, p-1$, where $\theta(t+\nu) = t + \nu \bmod p$.[3] In the case $p = 2$, $\varphi(t, 0, x)$ becomes $x_1 = A_0 x$, $x_2 = A_1 A_0 x$, $x_3 = A_0 A_1 A_0 x$, $x_4 = A_1 A_0 A_1 A_0 x$, ..., and $\varphi(t, 1, x)$ is given by $x_1 = A_1 x$, $x_2 = A_0 A_1 x$, $x_3 = A_1 A_0 A_1 x$, $x_4 = A_0 A_1 A_0 A_1 x$, ...

The theory of this type of difference equations is known as the Floquet theory. Excellent expositions can be found in Elaydi (2005, section 3,4) and Colonius and Kliemann (2014, section 7.1), among others. In the following, I will borrow freely from these expositions.

An immediate observation is that the nonautonomous equation (6.5) can be reduced to an autonomous one by taking p steps at once. For any solution $\{x_t\}$ define $y_\tau = x_{\tau p}$, $\tau \in \mathbb{Z}$. Then,

$$\begin{aligned}
x_{\tau p+1} &= A_{\tau p} x_{\tau p} = A_0 y_\tau, \\
x_{\tau p+2} &= A_{\tau p+1} x_{\tau p+1} = A_1 A_0 x_{\tau p} = A_1 A_0 y_\tau, \\
&\vdots \\
x_{\tau p+p} &= \left(\prod_{j=1}^{p} A_{p-j} \right) x_{\tau p} = \left(\prod_{j=1}^{p} A_{p-j} \right) y_\tau.
\end{aligned}$$

[3] If $t \in \mathbb{Z}$ then $t \bmod p$ denotes the unique integer r, $0 \le r < p$, such that $t = \tau p + r$ for some integer $\tau \in \mathbb{Z}$.

This shows that $\{y_\tau\}$ is a solution to the autonomous difference equation

$$y_{\tau+1} = \left(\prod_{j=1}^{p} A_{p-j} \right) y_\tau = \Phi(p) y_\tau$$

with $\Phi(p)$ given by equation (6.2). Given a solution $\{y_\tau\}$ determines a solution of equation (6.5) via the scheme from above: $x_{\tau p+1} = A_{\tau p} x_{\tau p} = A_0 y_\tau$ and so on.

As shown in Section 6.2, $\Phi(t)$ is a principal fundamental matrix. In addition, due to the periodicity of $\{A_t\}$, $\Phi(t)$ satisfies the relationship

$$\Phi(t + \tau p) = \Phi(t)\Phi(\tau p) = \Phi(t)\Phi(p)^\tau.$$

The periodicity of $\{A_t\}$ also translates into the periodicity of the *fundamental matrix* $\Phi(t, s)$ and Green's matrix $\Gamma(t, s)$:

$$\begin{aligned}
\Phi(t + \tau p, s + \tau p) &= \left(\Phi(t)\Phi(p)^\tau\right) \left(\Phi(s)\Phi(p)^\tau\right)^{-1} = \Phi(t, s) \\
\Gamma(t + \tau p, s + \tau p) &= \left(\Phi(t)\Phi(p)^\tau \Phi^{-1}(r)\right) \left(\Phi(s)\Phi(p)^\tau \Phi^{-1}(r)\right)^{-1} = \Gamma(t, s)
\end{aligned}$$

The asymptotic behavior of y_t and hence of x_t, thus depends on the eigenvalues α_j, $j = 1, \ldots, p$, of $\Phi(p)$, known as the *Floquet multipliers*. The *Floquet exponents* are defined as

$$\lambda_j = \frac{1}{p} \log |\alpha_j|.$$

It turns out that they play a crucial in understanding the asymptotic behavior of solutions of equation (6.5) and the stability of the zero solution. It is important to realize that the eigenvalues of the "time frozen" or "local" matrices A_t present no information with regard to the asymptotic behavior as the analysis and examples in Section 6.5.1 demonstrate.

The *Lyapunov exponent* of $\varphi(t, \nu, x)$ is defined as before by

$$\lambda(x, \nu) = \limsup_{t \to \infty} \frac{1}{t} \log \|\varphi(t, \nu, x)\| \qquad \text{for } (x, \nu) \in \mathbb{R}^d \times \{0, 1, \ldots, p-1\}.$$

In general the Lyapunov exponent may depend on ν. Given these preliminaries we can quote the following Theorem from Colonius and Kliemann (2014, theorem 7.1.7).

Theorem 6.3 (Floquet theory)**.** Consider the p-period linear difference equation (6.5). The Lyapunov exponents coincide with the Floquet exponents λ_j,

$j = 1, \ldots, \ell \leq d$, and they exist as a limit. For each $\nu \in \{0, 1, \ldots, p-1\}$ there exists a decomposition

$$\mathbb{R}^d = \mathbb{L}(\lambda_1, \nu) \oplus \cdots \oplus \mathbb{L}(\lambda_\ell, \nu)$$

into linear subspaces $\mathbb{L}(\lambda_j, \nu)$ called the *Floquet* or *Lyapunov spaces*. These subspaces have the following properties:

(i) The Lyapunov spaces have dimensions independent of ν,

$$d_j = \dim \mathbb{L}(\lambda_j, \nu) \text{ is constant for } \nu \in \{0, 1, \ldots, p-1\};$$

(ii) they are invariant under multiplication by the principal fundamental matrix in the following sense:

$$\Phi(t+\nu, \nu)\mathbb{L}(\lambda_j, \nu) = \mathbb{L}(\lambda_j, \theta(t, \nu)) \text{ for all } t \in \mathbb{Z} \text{ and } \nu \in \{0, 1, \ldots, p-1\}$$

where $\Phi(t + \nu, \nu)$ is defined as $\Phi(t + \nu)\Phi^{-1}(\nu)$;

(iii) for every $\nu \in \{0, 1, \ldots, p-1\}$, the Lyapunov exponent satisfy

$$\lambda(x, \nu) = \lim_{t \to \pm\infty} \frac{1}{t} \log \|\varphi(t, \nu, x)\| = \lambda_j$$

if and only if $x \in \mathbb{L}(\lambda_j, \nu)$ and $x \neq 0$.

Remark 6.2. The invariance property (ii) is known as equivariance. Note also that the Lyapunov spaces have periodicity p.

As in the constant coefficient case the Lyapunov subspaces can be collected into subbundles

$$\mathbb{L}^s(\nu) = \bigoplus_{\lambda_j < 0} \mathbb{L}(\lambda_j, \nu), \quad \mathbb{L}^c(\nu) = \mathbb{L}(0, \nu), \text{ and } \mathbb{L}^u(\nu) = \bigoplus_{\lambda_j > 0} \mathbb{L}(\lambda_j, \nu)$$

called the stable subbundle, the center, and the unstable subbundle, respectively. Thus, the zero solution is *asymptotically stable* if and only if all Lyapunov exponents are negative. This is equivalent to $\mathbb{L}^s(\nu) = \mathbb{R}^d$ for some (hence for all) $\nu \in \{0, 1, \ldots, p-1\}$. The difference equation (6.5) is called *hyperbolic* if $\mathbb{L}^c(\nu) = \emptyset$ or, equivalently, if all Lyapunov exponent are different from zero. For a hyperbolic difference equation the zero solution is called a *saddle point* if both $\mathbb{L}^s(\nu)$ and $\mathbb{L}^u(\nu)$ have dimensions $d^s = \dim \mathbb{L}^s(\nu)$, respectively $d^u = \dim \mathbb{L}^u(\nu)$, strictly greater than zero. From now on we assume the that the difference equation is hyperbolic.

Denote by $B(\nu)$ the matrix formed as the union of bases vectors $\mathbb{L}^s(\nu)$ and $\mathbb{L}^u(\nu)$. The assumption of hyperbolicity implies $\mathbb{R}^d = \mathbb{L}^s(\nu) \oplus \mathbb{L}^u(\nu)$ so that $B(\nu)$ is nonsingular. $B(\nu)$ and $B(\nu)^{-1}$ can be partitioned as

$$B(\nu) = \begin{pmatrix} B_{11}(\nu) & B_{12}(\nu) \\ B_{21}(\nu) & B_{22}(\nu) \end{pmatrix} \quad \text{and} \quad B^{-1}(\nu) = \begin{pmatrix} B^{11}(\nu) & B^{12}(\nu) \\ B^{21}(\nu) & B^{22}(\nu) \end{pmatrix}$$

such that $B_{11}(\nu)$ and $B^{11}(\nu)$ are invertible $d^s \times d^s$ matrices, and $B_{22}(\nu)$ and $B^{22}(\nu)$ are invertible $d^u \times d^u$ matrices. The projector $\pi^s(\nu) : \mathbb{R}^d \to \mathbb{L}^s(\nu)$ onto $\mathbb{L}^s(\nu)$ along $\mathbb{L}^u(\nu)$ is then

$$\pi^s(\nu) = B(\nu) \begin{pmatrix} I_{d^s} & 0 \\ 0 & 0 \end{pmatrix} B(\nu)^{-1}.$$

Similarly for the projector $\pi^u(\nu) : \mathbb{R}^d \to \mathbb{L}^u(\nu)$ onto $\mathbb{L}^u(\nu)$ along $\mathbb{L}^s(\nu)$ is given by

$$\pi^s(\nu) = B(\nu) \begin{pmatrix} 0 & 0 \\ 0 & I_{d^u} \end{pmatrix} B(\nu)^{-1}.$$

The periodicity of the Lyapunov spaces implies the periodicity of the corresponding projectors.

In economics, especially in the context of rational expectations models, we are often faced with a reversed boundary problem: Find an initial value x such that the solution $\varphi(t, \nu, x)$ of equation (6.5) does not explode, i.e. such that $\lambda(x, \nu) < 0$, subject to the restriction

$$c = Rx, \qquad c \neq 0 \text{ given}, \tag{6.6}$$

where R is a $(r \times d)$–matrix of rank r. Depending on the rank r one can distinguish several cases:

$r = 0$: the requirement (6.6) places no restriction. In this situation $x_t = \varphi(t, \nu, 0) = 0$ for all t is the unique non–explosive solution if and only if $\mathbb{L}^s(\nu) = \{0\}$ or equivalently if $d^s = \dim \mathbb{L}^s(\nu) = 0$, respectively $d^u = \dim \mathbb{L}^u(\nu) = d$.

$r = d$: The condition (6.6) determines a unique initial value $x = R^{-1}c$. If this x lies in $\mathbb{L}^s(\nu)$ then $x_t = \varphi(t, \nu, x)$ is the unique non–explosive solution. This is obviously the case if $\mathbb{L}^s(\nu) = \mathbb{R}^d$. Otherwise no non–explosive solution exists.

$0 < r < d$: In this case x is determined by a simultaneous equation system consisting of the restriction (6.6) and the condition $\pi^s(\nu)x = x$, respectively $\pi^u(\nu)x = 0$. The last condition simplifies t $(0, I_{d^u})B^{-1}(\nu)x = 0$

or $(B^{21}, B^{22})x = 0$. Hence, a unique non–explosive solution is obtained if and only if

$$\operatorname{rank}\begin{pmatrix} R \\ \begin{pmatrix} 0 & I_{d^u} \end{pmatrix} B(\nu)^{-1} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} R \\ B^{(2)}(\nu) \end{pmatrix} = d. \tag{6.7}$$

where $B^{(2)}(\nu) = (B^{21}(\nu), B^{22}(\nu))$. If this equation system has a unique solution, the difference equation (**??**) is said to be *determinate*. As R has r rows and $\begin{pmatrix} 0 & I_{d^u} \end{pmatrix} B^{-1}(\nu)$ has $d^u = d - d^s$ rows, a necessary condition for a unique non–explosive solution is that $r = d^s$. If $r < d^s$, there is a whole family of solutions and the system (6.5) is then called *indeterminate*. If $r > d^s$, the equation system is overdetermined and no solution exists.

Finding a Particular Solution Having completely characterized the linear part and recognizing that the superposition principle holds, we need to find a particular solution to solve the affine equation system (6.4).

Theorem 6.4 (Solution Periodic Coeficients)**.** The boundary value problem consisting of the hyperbolic difference equation (6.4) subject to the initial condition(6.6), and the boundedness condition admits a unique solution of the form

$$x_t = \Phi(t)x + x_t^{(b)} + x_t^{(f)}$$

where

$$x_t^{(b)} = \Phi(t, \nu) \sum_{j=0}^{\infty} \Phi(t - j, \nu)^{-1} \pi^s(\nu) b_{t-1-j}$$
$$x_t^{(f)} = -\Phi(t, \nu) \sum_{j=0}^{\infty} \Phi(t + j + 1, \nu)^{-1} \pi^s(\nu) b_{t+j}$$

if and only if the rank condition (6.7) is satisfied.

Proof. □

Remark 6.3. In order not to overload the notation, the dependence of x_t, $x_t^{(b)}$, and $x_t^{(f)}$ on ν is not explicitly mentioned.

6.5 Examples of Periodically Time-Varying Coefficient Models

6.5.1 Uninformativeness of Eigenvalues

This section provides a systematic way to construct time-varying (nonautonomous) difference equations such that every coefficient matrix would imply stability whereas the system actually diverges.[4] Consider the deterministic two-dimensional system

$$x_{t+1} = A_t x_t \qquad \text{in } \mathbb{R}^2$$

where $A_t = \exp(tG(\omega))B\exp(-tG(\omega))$, $\omega > 0$, with

$$G(\omega) = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \quad \text{implying } \exp(tG(\omega)) = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}$$

Because A_t and B are similar matrices, they share the same eigenvalues. Define $y_t = \exp(-tG(\omega))x_t$. Thus, y_t is obtained from x_t by a rotation with angle ωt. Using the defining difference equation for x_t, we see that y_t follows the autonomous difference equation

$$\begin{aligned} y_{t+1} &= \exp(-(t+1)G(\omega))x_{t+1} \\ &= \exp(-(t+1)G(\omega))\underbrace{\exp(tG(\omega))B\exp(-tG(\omega))}_{=A_t}x_t \\ &= \exp(-G(\omega))By_t. \end{aligned}$$

Thus, x_t diverges if and only if y_t diverges. The stability of y_t is determined by the matrices $\exp(-G(\omega))$ and B. If we can find a matrix B and an ω such that $\rho(\exp(-G(\omega))B) > 1$ and $\rho(B) < 1$, we have found an example where each of the "time frozen" coefficient matrices would imply stability, but where the nonautonomous system is unstable.

One such specification inspired by Elaydi (2005, p. 190) is obtained by taking

$$\omega = 1 \quad \text{and } B = \begin{pmatrix} 0 & 1/2 \\ 3/2 & 0 \end{pmatrix}.$$

In this example the eigenvalues of B are $\pm\sqrt{3}/2$, thus both smaller than one in absolute terms, but $\exp(-G(\omega))B$ has eigenvalues $1.3836 > 1$ and

[4]This construction translates the continuous time approach of Colonius and Kliemann (2014, p.109–110) to a discrete time framework. Francq and Zakoïan (2001) provide another ad hoc examples in a time series context.

-0.5421. x_t therefore diverges although every A_t has eigenvalues with modulus strictly smaller than one.

Another specification is obtained by taking

$$\omega = 2 \quad \text{and } B = \begin{pmatrix} 1/10 & 1/2 \\ -3/2 & 1/10 \end{pmatrix}.$$

In this case the eigenvalues of B are $0.1 \pm \imath\sqrt{3}/2$ whose moduli are strictly smaller than one. The eigenvalues of $\exp(-G(\omega))B$, however, are 1.3307 and -0.5711. Thus again, x_t diverges although every A_t has eigenvalues with modulus strictly smaller than one.

6.5.2 A Numerical Example

To get a better understanding of the behavior of periodically switching linear difference equations, we consider the following numerical example for $d = 2$ and $p = 2$:

$$A_0 = \begin{pmatrix} 1 & 0.2 \\ 1 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & -0.5 \\ 3 & -2 \end{pmatrix}.$$

These matrices have eigenvalues $\lambda_1^{(0)} = 1.4472$ and $\lambda_2^{(0)} = 0.5528$, respectively $\lambda_1^{(0)} = 0.3660$ and $\lambda_2^{(0)} = -1.3660$. The matrices A_1A_0 and A_0A_1 are then given by

$$A_1A_0 = \begin{pmatrix} 0.5 & -0.3 \\ 1.0 & -1.4 \end{pmatrix}, \qquad A_0A_1 = \begin{pmatrix} 1.6 & -0.9 \\ 4.0 & -2.5 \end{pmatrix}.$$

The Lyapunov, respectively the Floquet, exponents in this case can be computed from the eigenvalues α_1 and α_2 of A_1A_0 and A_0A_1 as $\lambda_j = \frac{1}{2}\log|\alpha_j|$, $j = 1, 2$. This gives $\lambda_1 = -0.5601$ and $\lambda_2 = 0.1020$. From the eigenvectors of A_1A_0 and A_0A_1 we can compute the stable and the unstable bundle:

$$L^s(0) = \operatorname{span}\begin{pmatrix} 0.8653 \\ 0.5013 \end{pmatrix} \qquad L^u(0) = \operatorname{span}\begin{pmatrix} 0.1712 \\ 0.9852 \end{pmatrix}$$

and

$$L^s(1) = \operatorname{span}\begin{pmatrix} 0.5770 \\ 0.8167 \end{pmatrix} \qquad L^u(1) = \operatorname{span}\begin{pmatrix} 0.3034 \\ 0.9529 \end{pmatrix}.$$

The corresponding projector matrices are

$$\pi^s(0) = \begin{pmatrix} 1.1119 & -0.1932 \\ 0.6442 & -0.1119 \end{pmatrix} \qquad \pi^u(0) = \begin{pmatrix} -0.1119 & 0.1932 \\ -0.6442 & 1.1119 \end{pmatrix}$$

and

$$\pi^s(1) = \begin{pmatrix} 1.8205 & -0.5797 \\ 2.5766 & -0.8205 \end{pmatrix} \qquad \pi^u(1) = \begin{pmatrix} -0.8205 & 0.5797 \\ -2.5766 & 1.8205 \end{pmatrix}.$$

6.5.3 The New Keynesian Model with Periodically Switching Policy Rules

As an example consider a simple version of the New Keynesian macroeconomic model introduced and analyzed by Galí (2011). The model consists of the following three equations:

$$\begin{aligned} y_t &= y_{t+1} - \frac{1}{\sigma}(i_t - \pi_{t+1}), && \text{(IS-equation)} \\ \pi_t &= \beta\pi_{t+1} + \kappa y_t + u_t, && \text{(forward-looking Phillips-curve)} \\ i_t &= \phi\pi_t, && \text{(Taylor-rule)} \end{aligned}$$

where y_t, π_t, and i_t denote income, inflation and the nominal interest rate, all measured as deviations from the steady state. u_t is an exogenous cost-push shock. Furthermore, we assume that $\sigma > 0$, $\kappa > 0$, and $0 < \beta \leq 1$. In addition, $\phi > 0$ measures how aggressive the central bank is in combatting inflation.

This model can be solved for $(y_{t+1}, \pi_{t+1})'$ by inserting the Taylor-rule and Phillips-curve into the IS-equation:

$$\begin{aligned} x_{t+1} = \begin{pmatrix} \pi_{t+1} \\ y_{t+1} \end{pmatrix} &= \frac{1}{\beta} \begin{pmatrix} 1 & -\kappa \\ (\phi\beta - 1)/\sigma & \beta + \kappa/\sigma \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} + \begin{pmatrix} -u_t/\beta \\ u_t/(\sigma\beta) \end{pmatrix} \\ &= A_0 x_t + b_{0,t} \end{aligned} \tag{6.8}$$

Denote the characteristic polynomial of A_0 by $\mathcal{P}(\lambda)$ and the corresponding eigenvalues by λ_1 and λ_2, then we have

$$\mathcal{P}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - \operatorname{tr}(A_0)\lambda + \det A_0$$

with

$$\begin{aligned} \operatorname{tr} A_0 &= \lambda_1 + \lambda_2 = 1 + \frac{1}{\beta} + \frac{\kappa}{\sigma\beta} > 2 \\ \det A_0 &= \lambda_1\lambda_2 = \frac{1}{\beta} + \frac{\kappa\phi}{\sigma\beta} > 1 \\ \Delta_0 &= (\operatorname{tr} A_0)^2 - 4\det A_0 = \left(1 - \frac{1}{\beta}\right)^2 + \frac{\kappa}{\sigma\beta}\left(\frac{\kappa}{\sigma\beta} + 2 + \frac{2}{\beta} - 4\phi\right) \\ \mathcal{P}(1) &= (1 - \lambda_1)(1 - \lambda_2) = \frac{\kappa}{\sigma\beta}(\phi - 1) > 0, \quad \text{if } \phi > 1, \end{aligned}$$

where Δ_0 denotes the discriminant of the quadratic equation. Depending on ϕ, the roots of $\mathcal{P}(\lambda)$ may be complex. We therefore distinguish two cases. First assume that ϕ is high such that $\Delta_0 < 0$. In this case we

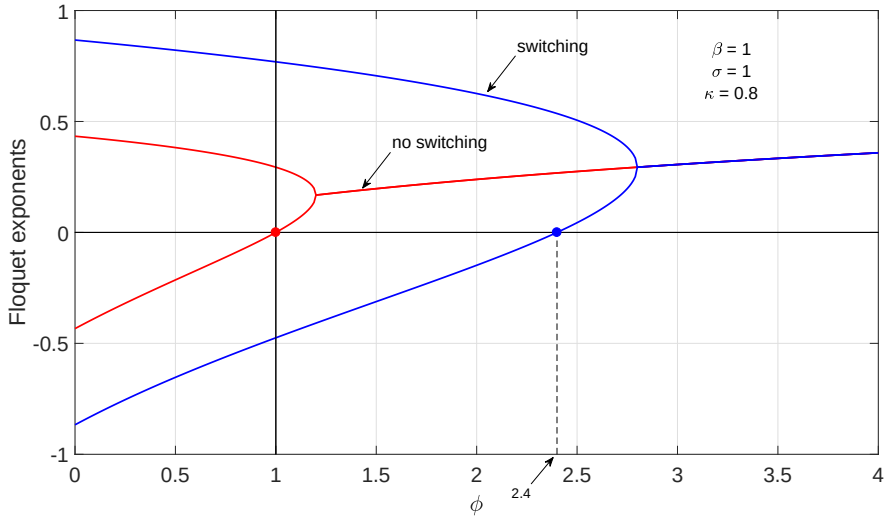

Figure 6.2: The Dynamics of the New Keynesian Model with and without Switching

have two complex conjugate roots. Because $\det A_0 > 1$, they are located outside the unit circle.[5] Alternatively assume that ϕ is small enough such that $\Delta_0 > 0$. In this case both eigenvalues are real. Using the assumption $\phi > 1$, $\mathcal{P}(1) > 0$. Thus, both roots are either greater or smaller than one. They cannot be smaller than one because $\mathrm{tr} A_0 > 2$. Thus, in both cases we reach the conclusion that the eigenvalues are outside the unit circle. When ϕ is smaller than one, we have two real positive roots: one smaller than one, the other bigger than one. The situation with $\sigma = \beta = 1$ and $\kappa = 0.8$ is illustrated in Figure 6.2 by the red curve which shows the modulus of the two eigenvalues depending on the value of ϕ.

As both variables are non-predetermined, the boundedness condition then determines the unique solution:

$$x_t = \sum_{j=1}^{\infty} A_0^{-j} b_{0,t+j-1} = \sum_{j=1}^{\infty} Q \begin{pmatrix} \lambda_1^{-j} & 0 \\ 0 & \lambda_2^{-j} \end{pmatrix} Q^{-1} \begin{pmatrix} u_{t-1+j}/\beta \\ -u_{t-1+j}/(\sigma\beta) \end{pmatrix}$$

where the columns of Q consist of the eigenvectors corresponding to λ_1 and λ_2.

Suppose that the central bank changes its policy and fixes the path of the interest rate. The interest rate then becomes an exogenous variable and the

[5] Another way to reach this conclusion is by observing that the real part of the roots is $\frac{\mathrm{tr} A_0}{2} > 1$.

system changes to:

$$\begin{aligned} x_{t+1} = \begin{pmatrix} \pi_{t+1} \\ y_{t+1} \end{pmatrix} &= \frac{1}{\beta} \begin{pmatrix} 1 & -\kappa \\ -1/\sigma & \beta + \kappa/\sigma \end{pmatrix} \begin{pmatrix} \pi_t \\ y_t \end{pmatrix} + \begin{pmatrix} -u_t/\beta \\ i_t^*/\sigma + u_t/(\sigma\beta) \end{pmatrix} \\ &= A_1 x_t + b_{1,t} \end{aligned} \tag{6.9}$$

where i_t^* is the exogenous path of the interest rate. The trace, the determinant, and the discriminant, Δ_1, of the characteristic polynomial then become:

$$\begin{aligned} \mathrm{tr} A_1 &= \lambda_1 + \lambda_2 = 1 + \frac{1}{\beta} + \frac{\kappa}{\sigma\beta} > 2 \\ \det A_1 &= \lambda_1 \lambda_2 = \frac{1}{\beta} > 1 \\ \Delta_1 &= (\mathrm{tr} A_1)^2 - 4 \det A_1 = \left(1 - \frac{1}{\beta}\right)^2 + \frac{\kappa}{\sigma\beta}\left(\frac{\kappa}{\sigma\beta} + 2 + \frac{2}{\beta}\right) > 0 \\ \mathcal{P}(1) &= (1 - \lambda_1)(1 - \lambda_2) = -\frac{\kappa}{\sigma\beta} < 0. \end{aligned}$$

The discriminant Δ_1 becomes unambiguously positive implying that both eigenvalues are real. Moreover, $\mathcal{P}(1) < 0$ so that one eigenvalue is smaller than one whereas the other bigger than one. Thus, the boundedness condition does not determine a unique solution so that we are faced with a situation of indeterminacy. The implications of this indeterminacy for monetary policy and possible remedies are discussed in Galí (2011). Taking the same numerical values as before, i.e. $\sigma = \beta = 1$ and $\kappa = 0.8$, we get $\lambda_1 = 0.42$ and $\lambda_2 = 2.38$.

Suppose next that the central bank switches deterministically between the two policies starting with the model with Taylor rule followed by the model without Taylor rule. This and similar settings are discussed in Galí (2011, section 4.1.1) and, in particular, **?**. As shown in Figure 6.2 there is still a region of the parameter space where the model becomes determinate. However, the central bank must be much more aggressive in combatting inflation. In the numerical example ϕ must be greater than 2.4 instead of one.

6.6 Deterministically Time-Varying Coefficients

deterministic MET

6.7 Randomly Time-Varying Coefficients

Oseledets Theorem

6.8 Examples of Randomly Time-Varying Coefficient Models

6.8.1 Fractal Dimension of Random Coefficient Models

A surprising feature of random affine transformations is their ability to draw two-dimensional pictures. Consider the following example from Berger (1993, p.164):

$$x_{t+1} = A_t x_t + b_t$$

and (A_t, b_t) is chosen randomly from the set $\{(A^{(i)}, b^{(i)})\}$, $i = 1, 2$, where the corresponding matrices and vectors are specified as

$$A_1 = \begin{pmatrix} 0.183953 & 0.0 \\ 0.0 & 0.1846053 \end{pmatrix} \qquad b_1 = \begin{pmatrix} 0.7331354 \\ 0.4357292 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 0.8728180 & -0.4115444 \\ 0.4115444 & 0.8728180 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 0.2725137 \\ -0.1435450 \end{pmatrix}.$$

The configurations $\{(A^{(i)}, b^{(i)})\}$ are chosen independently with probabilities $p_1 = 0.06982$ and $p_2 = 1 - p_1 = 0.93018$. A simulation with $10'000$ replications results in the scatter plot 6.3. Note the fractal geometry of this picture. This is not a coincidence as emphasized in Berger (1993) where many other examples are presented.

An interesting aspect of this type of models is that it is possible to start from a picture which is taken as the stationary distribution of the random process. The corresponding parameters are then computed using the ergodicity of the process. See Diaconis and Freedman (1999, section 2.3) for more details and references.

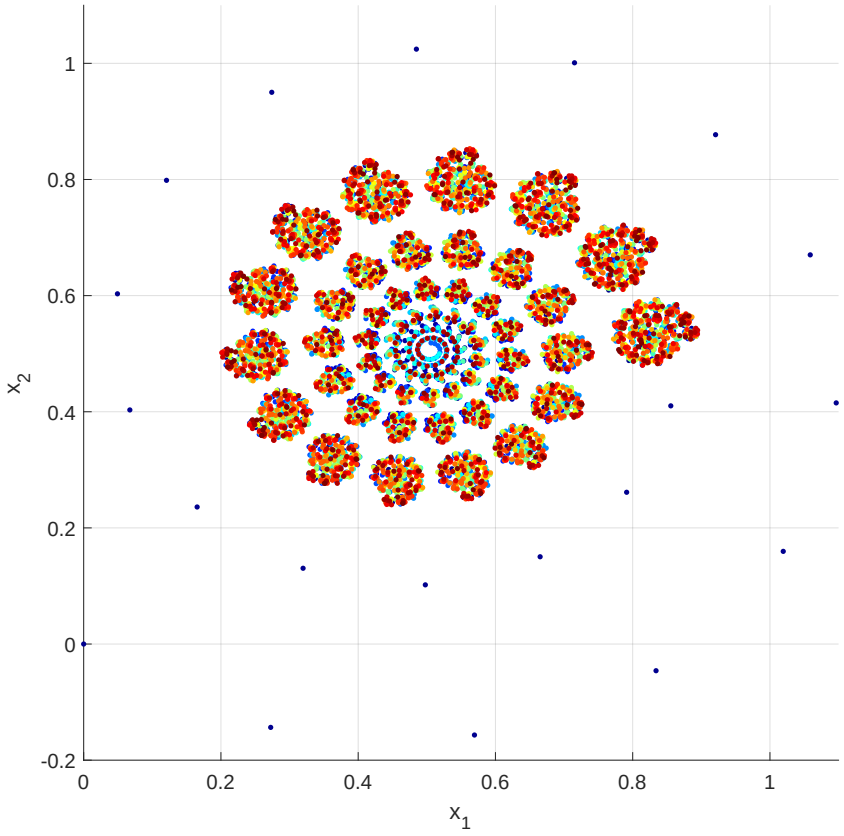

Figure 6.3: Fractal Dimension of Randomly Varying Affine Transformations

Appendix A

Elements of Calculus

A.1 Prerequisites for one Variable Functions

This section summarizes some concepts and results from elementary calculus pertaining real valued functions in one variable.

For any differentiable function f a point x with the property $f'(x) = 0$ is called a *critical point*. If in addition $f''(x) \neq 0$ the point is said to be a *nondegenerate critical point*. For example,the logistic function (1.8) has $x = 1/2$ as a nondegenerate critical point. If $f(x) = x^3$ then $x = 0$ is a degenerate critical point.

Theorem A.1 (Intermediate Value Theorem)**.** Let $f : I \to \mathbb{R}$ be a continuous function where I is an interval.[1] Take any value y between $f(a)$ and $f(b)$ where $a, b \in I$ and $a < b$ then there exists at least one $x \in I$ with $a < x < b$ such that $y = f(x)$.

This theorem says that f attains all the values between $f(a)$ and $f(b)$; or, to put it otherwise, $f([a, b])$ is connected.

Theorem A.2 (Mean Value Theorem)**.** Let $f : I \to \mathbb{R}$ be a continuous function where I is an interval. For any values $a, b \in I$ with $a < b$ and f differentiable on (a, b) there exists $x \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(x).$$

Hence there exists a point x at which the derivative is equal to the slope of the secant line from $(a, f(a))$ to $(b, f(b))$.

[1] It does not matter whether I is open, closed, or half-open.

As the dynamics is mostly generated by iterating a given function, it is instructive to revisit the chain rule of differentiation and some of its implications. Let $f, g : \mathbb{R} \to \mathbb{R}$ two differentiable function then for their composition denoted by $\circ$ we have

$$(g \circ f)'(x) = g'(f(x))f'(x) = g'(y)f'(x)$$

where $y = f(x)$. Note that the derivative of g must be taken at the correct point y.

The iteration of a given function f starting in x_0 leads to $x_1 = f(x_0)$, $x_2 = f(x_1)$, and $x_t = f(x_{t-1})$, $t = 1, 2, \ldots$ Thus, $x_t = \underbrace{f \circ \cdots \circ f}_{t-\text{fold}}(x)$ denoted by $x_t = f^t(x)$. The application of the chain rule leads to

$$(f^t)'(x) = f'(x_{t-1}) \ldots f'(x_1)f'(x_0).$$

The point here is that we do not need to compute f^t to calculate $(f^t)'$. This is important because the explicit computation of f^t can quickly become intractable or impossible, especially for large t. An only three-fold iteration of the logistic function (1.8), for example, makes f^3 already a polynomial of degree 6. For $\mu = 1$, starting at $x_0 = 1/4$ leads to $x_1 = 3/16$ and $x_2 = 39/256$. As $f'(x) = (1 - 2x)$, the derivative of f^3 at $x = 1/4$ can be computed as follows.

$$\begin{aligned}(f^3)'(1/4) = f'(39/256)f'(3/16)f'(1/4) &= \left(1 - \frac{78}{256}\right)\left(1 - \frac{6}{16}\right)\left(1 - \frac{2}{4}\right) \\ &= \frac{178}{256}\,\frac{10}{16}\,\frac{2}{4}\end{aligned}$$

A.2 Differentiation in Higher Dimensions

Definition A.1. Suppose E is an open set in $\mathbb{R}^n$, F maps E into $\mathbb{R}^m$, and $x \in E$. If there exists a linear transformation A from $\mathbb{R}^n$ into $\mathbb{R}^m$ such

$$\lim_{h \to 0} \frac{\|F(x + h) - F(x) - Ah\|}{\|h\|} = 0,$$

then we say that F is differentiable at x, and we write

$$F'(x) = A, \qquad \mathrm{d}F(x) = F(x + h) - F(x) = A\mathrm{d}x$$

A is called the *derivative* or *total derivative*.

Definition A.2. A differentiable mapping F from an open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$ is called *continuously differentiable* in E if the linear transformation F' from $\mathbb{R}^n$ into $\mathbb{R}^m$ is continuous on E. We write $F \in \mathcal{C}^1(E)$.

Theorem A.3. Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, f is differentiable at $x_0 \in E$, g maps an open set containing $f(E)$ into $\mathbb{R}^k$, and g is differentiable at $f(x_0)$. Then the mapping F of E into $\mathbb{R}^k$ defined by $F(x) = (g \circ f)(x) = g(f(x))$ is differentiable at x_0, and

$$F'(x_0) = g'(f(x_0))f'(x_0)$$

Consider again a function F from an open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$ and let $\{e_1, \ldots, e_n\}$ and $\{u_1, \ldots, u_m\}$ be the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. The components of F are the function $f_1, \ldots, f_m$ defined by

$$F(x) = \sum_{i=1}^{m} f_i(x)u_i, \qquad x \in E.$$

The *partial derivative* of f_i with respect to x_j, keeping the other variables fixed, is defined as

$$(D_j f_i)(x) = \frac{\partial f_i(x)}{\partial x_j} = \lim_{t \to 0} \frac{f_i(x + te_j) - f_i(x)}{t}$$

Theorem A.4. Suppose that F is differentiable at point x, then the partial derivatives exist and we have

$$F'(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{pmatrix}$$

This matrix is called the *Jacobian matrix*. The converse is not true. In particular, we have

$$F'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x)u_i, \qquad 1 \leq j \leq n.$$

Theorem A.5. Suppose $F : E \subset \mathbb{R}^n \to \mathbb{R}^m$, E open, such $F(x) = (f_1(x), \ldots, f_m(x))'$. Then, $F \in \mathcal{C}^1(E)$ if and only if the partial derivatives $D_j f_i = \frac{\partial f_i(x)}{\partial x_j}$ exist and are continuous for $1 \leq i \leq m$ and $1 \leq j \leq m$.

Thus, we have

$$\left\{\begin{matrix}\text{partial derivatives}\\ \text{are continuous}\end{matrix}\right\} \Rightarrow \text{derivative of } F \text{ exists}$$

$$\left\{\begin{matrix}\Rightarrow F \text{ is continuous}\\ \Rightarrow \text{partial derivatives exist}\end{matrix}\right\}$$

Let $F : E \subset \mathbb{R}^n \to \mathbb{R}$, E open, with $F \in \mathcal{C}^2(E)$ and define $\frac{\partial}{\partial x_j}\left(\frac{\partial F(x)}{\partial x_i}\right) = \frac{\partial^2 F(x)}{\partial x_j \partial x_i}$. Arranged in an $n \times n$ matrix, we get the *Hessian matrix*:

$$D^2F_x = \begin{pmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_1} \\ \frac{\partial^2 F}{\partial x_1 \partial x_2} & \frac{\partial^2 F}{\partial x_2^2} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_1 \partial x_n} & \frac{\partial^2 F}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 F}{\partial x_n^2} \end{pmatrix}$$

The Hessian matrix is symmetric because

$$\frac{\partial^2 F}{\partial x_j \partial x_i} = \frac{\partial^2 F}{\partial x_i \partial x_j}$$

as long as $F \in \mathcal{C}^2(E)$.

Definition A.3. Let $F : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function at x^*. The derivative viewed as vector of the partial derivatives is called the *gradient vector* of F at x^* and denoted by $(\nabla F)(x^*)$:

$$(\nabla F)(x^*) = \begin{pmatrix} \frac{\partial F}{\partial x_1}(x^*) \\ \vdots \\ \frac{\partial F}{\partial x_n}(x^*) \end{pmatrix} = \sum_{i=1}^{n} (D_i F)(x) e_i.$$

Definition A.4. Let u be a unit vector (i.e. $\|u\| = 1$) of $\mathbb{R}^n$, then

$$\lim_{h \to 0} \frac{F(x + hu) - F(x)}{h} = \langle (\nabla F)(x), u \rangle = \|(\nabla F)(x)\| \cos\theta$$

is called the *directional derivative* of F at x in the direction of u.

Theorem A.6. Let $F : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a $\mathcal{C}^1(U)$ function. At any point $x \in U$ at which $\nabla F(x) \neq 0$, the gradient vector points at x into the direction in which F increases most rapidly.

Theorem A.7 (Implicit Function Theorem)**.** Let $F(y,x) \in \mathcal{C}^1(E)$ with $E \subseteq \mathbb{R}^{n+m}$, E open, into $\mathbb{R}^n$ such that $F(y_0,x_0) = 0$ for some point $(y_0,x_0) \in E$. Put $A = F'(y_0,x_0)$ and assume that A_y is invertible. Then there exists open sets $U \subseteq \mathbb{R}^{n+m}$ and $W \subseteq \mathbb{R}^m$, with $(y,x) \in U$ and $x \in W$ having the following properties:

- To every $x \in W$ corresponds a unique y such that

$$(y,x) \in U \quad \text{and} \quad F(y,x) = 0.$$

- If this y is defined to be $y = G(x)$, then $G \in \mathcal{C}^1(W)$ into $\mathbb{R}^n$, $G(x_0) = y_0$, $F(G(x),x) = 0$ for $x \in W$.
- $G'(x_0) = -(A_y)^{-1}A_x$.

Appendix B

Complex Numbers

As the simple quadratic equation $x^2 + 1 = 0$ has no solution in the field of real numbers, $\mathbb{R}$, it is necessary to envisage the larger field of complex numbers $\mathbb{C}$. A complex number z is an ordered pair (a, b) of real numbers where ordered means that we regard (a, b) and (b, a) as distinct if $a \neq b$. We endow the set of complex numbers by an addition and a multiplication. Let $x = (a, b)$ and $y = (c, d)$ be two complex numbers, then we have the following definitions:

$$\begin{aligned} &\text{addition:} && x + y = (a, b) + (c, d) = (a + c, b + d) \\ &\text{multiplication:} && xy = (a, b)(c, d) = (ac - bd, ad + bc). \end{aligned}$$

These two operations will turn $\mathbb{C}$ into a field where $(0, 0)$ and $(1, 0)$ play the role of 0 and 1.[1] The real numbers $\mathbb{R}$ are embedded into $\mathbb{C}$ because we identify any $a \in \mathbb{R}$ with $(a, 0) \in \mathbb{C}$.

The number $\imath = (0, 1)$ is of special interest. It solves the equation $x^2 + 1 = 0$, i.e. $\imath^2 = -1$. The other solution being $-\imath = (0, -1)$. Thus any complex number $z = (a, b)$ may be written alternatively as $z = (a, b) = a + \imath b$ where a, b are arbitrary real numbers.[2]

[1] Substraction and division can be defined accordingly:

$$\begin{aligned} &\text{subtraction:} && (a, b) - (c, d) = (a - c, b - d) \\ &\text{division:} && (a, b)/(c, d) = \frac{(ac + bd, bc - ad)}{(c^2 + d^2)}, \quad c^2 + d^2 \neq 0. \end{aligned}$$

[2] A more detailed introduction of complex numbers can be found in Rudin (1976) or any other mathematics textbook.

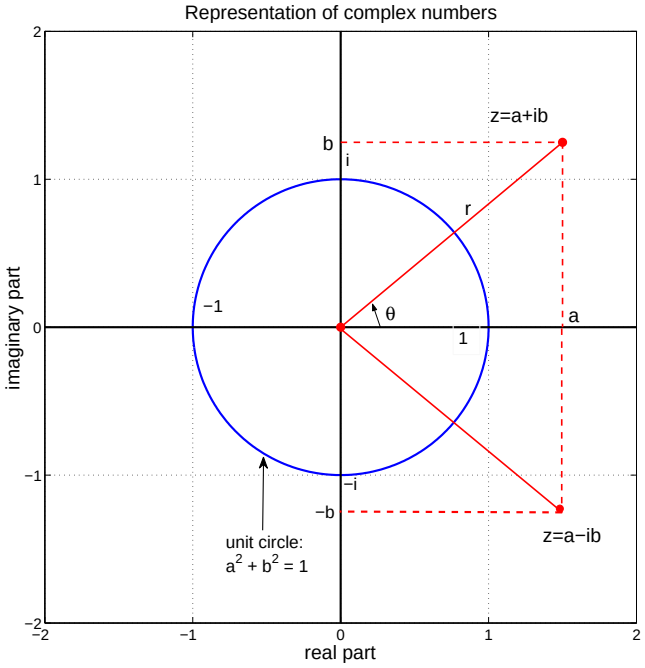

Figure B.1: Representation of a complex number

An element $z \in \mathbb{C}$ can be represented in two ways:

$$\begin{aligned} z &= a + \imath b && \text{Cartesian coordinates} \\ &= re^{\imath\theta} = r(\cos\theta + \imath \sin\theta) && \text{polar coordinates.} \end{aligned}$$

In the representation in Cartesian coordinates $a = \mathrm{Re}(z) = \Re(z)$ is called the *real part* whereas $b = \mathrm{Im}(z) = \Im(z)$ is called the *imaginary part* of z.

A complex number z can be viewed as a point in the two-dimensional Cartesian coordinate system with coordinates (a, b). This geometric interpretation is represented in Figure B.1.

The *absolute value* or *modulus* of z, denoted by $|z|$, is given by $r = \sqrt{a^2 + b^2}$. Thus the absolute value is nothing but the Euclidean distance of z viewed as a point in the complex plane (the two-dimensional Cartesian coordinate system) to the origin (see Figure B.1). θ denotes the angle to the positive real axis (x-axis) measured in radians. It is denoted by $\theta = \arg z$. It holds that $\tan\theta = \frac{b}{a}$. Finally, the *conjugate* of z, denoted by $\bar{z}$, is defined by $\bar{z} = a - \imath b$.

Setting $r = 1$ and $\theta = \pi$, gives the following famous formula:

$$e^{\imath\pi} + 1 = (\cos\pi + \imath \sin\pi) + 1 = -1 + 1 = 0.$$

This formula relates the most famous numbers in mathematics.

From the definition of complex numbers in polar coordinates, we get immediately the following implications:

$$\cos\theta = \frac{e^{\imath\theta} + e^{-\imath\theta}}{2} = \frac{a}{r},$$
$$\sin\theta = \frac{e^{\imath\theta} - e^{-\imath\theta}}{2\imath} = \frac{b}{r}.$$

Further implications are de Moivre's formula and the Pythagoras' theorem (see Figure B.1):

$$\begin{aligned} &\text{de Moivre's formula} && \left(re^{\imath\theta}\right)^n = r^n e^{\imath n\theta} = r^n(\cos n\theta + \imath \sin n\theta) \\ &\text{Pythagoras' theorem} && 1 = e^{\imath\theta}e^{-\imath\theta} = (\cos\theta + \imath\sin\theta)(\cos\theta - \imath\sin\theta) \\ & && = \cos^2\theta + \sin^2\theta \end{aligned}$$

From Pythagoras' theorem it follows that $r^2 = a^2 + b^2$. The representation in polar coordinates allows to derive many trigonometric formulas.

Consider the polynomial $\mathcal{P}(z) = \phi_0 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p$ of order $p \geq 1$ with $\phi_0 = 1$.[3] The *fundamental theorem of algebra* then states that every polynomial of order $p \geq 1$ has exactly p roots in the field of complex numbers. Thus, the field of complex numbers is algebraically complete. Denote these roots by $\lambda_1, \ldots, \lambda_p$, allowing that some roots may appear several times. The polynomial can then be factorized as follows:

$$\mathcal{P}(z) = \left(1 - \lambda_1^{-1}z\right)\left(1 - \lambda_2^{-1}z\right)\ldots\left(1 - \lambda_p^{-1}z\right).$$

This expression is well-defined because the assumption of a nonzero constant ($\phi_0 = 1 \neq 0$) excludes the possibility of roots equal to zero. If we assume that the coefficients ϕ_j, $j = 0, \ldots, p$, are real numbers, the roots appear in conjugate pairs. Thus if $z = a + \imath b$, $b \neq 0$, is a root then $\bar{z} = a - \imath b$ is also a root.

[3]The notation with "$-\phi_j z^j$" instead of "$\phi_j z^j$" was chosen to conform to the notation of auto regressive models.

Appendix C

Concepts of Linear Algebra

This appendix summarizes the concepts and results from linear algebra which are relevant for this book. For details and proofs the reader is referred to standard textbooks (see f.e. Meyer (2000) or Strang (2003)).

C.1 Algebraic and Topological Structure

The state space $\mathbf{X}$ is endowed with an algebraic and a topological structure. The algebraic structure is the one of *vector* or *linear* space over the real number $\mathbb{R}$. This means that $\mathbf{X}$ is closed under the addition operation $+$ and the multiplication of a vector by a scalar. More precisely, $\alpha x + \beta y \in \mathbf{X}$ for all $x, y \in \mathbf{X}$ and all $\alpha, \beta \in \mathbb{R}$. The topological structure is induced by a *norm*. A norm associates to each vector $x \in \mathbf{X}$ a positive real number $\|x\|$ with the following properties:

(i) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in \mathbf{X}$ and all $\alpha \in \mathbb{R}$.

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbf{X}$. This property is called the *triangle inequality*.

$\|x\|$ can be interpreted as the length of the vector. A vector space with a norm is called a *normed space*. Every normed space is also a *topological* space allowing for the definition of openness, closeness, and continuity.

A metric space is a vector space with a *distance* or *metric*. A distance is a function $d(x, y) : \mathbf{X} \times \mathbf{X} \to [0, \infty)$ such that for all $x, y, z \in \mathbf{X}$:

(i) nonnegativity: $d(x, y) \geq 0$.

(ii) identity of indiscernibles: $d(x, y) = 0$ implies $x = y$.

(iii) symmetry: $d(x, y) = d(y, x)$.

(iv) triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

A norm induces a metric by defining $d(x, y) = \|x-y\|$. Hence, normed spaces are a subset of metric spaces.

Most of the time $\mathbf{X}$ is identified with the d-dimensional Euclidean space $\mathbb{R}^d$. The corresponding Euclidean norm for some vector $x = (x_1, \ldots, x_d)$ is defined as $\|x\| = \sqrt{x_1^2 + \ldots + x_d^2}$.

Two vectors x and y are called *linearly dependent* if there exists $\alpha, \beta \in \mathbb{R}$, not both equal to zero, such that $\alpha x + \beta y = 0$, otherwise they are called *linearly independent*.

C.2 Linear Transformations

A *linear transformation* is a function $A : \mathbf{X} \to \mathbf{X}$ such that

(i) additivity: $A(x + y) = Ax + Ay$ for all $x, y, \in \mathbf{X}$.

(ii) homogeneity: $A(\alpha x) = \alpha Ax$ for all $x \in \mathbf{X}$ and all $\alpha \in \mathbb{R}$.

If $\mathbf{X}$ is identified with the d-dimensional Euclidean space $\mathbb{R}^d$, every linear transformation can be represented as a two-dimensional array of numbers, called a matrix:[1]

$$A = (A)_{i,j} = \begin{pmatrix} a_{11} & \ldots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \ldots & a_{dd} \end{pmatrix}$$

The set of all $d \times d$ matrices is denoted by $\mathbb{M}(d)$.

Every matrix defines four fundamental vector (sub)spaces:

(i) column space $\mathbf{R}(A) = \{y \in \mathbb{R}^d : \exists x \in \mathbb{R}^d \text{ with } y = Ax\}$

(ii) null space or kernel, $\mathbf{N}(A) = \{x \in \mathbb{R}^d : Ax = 0\}$

(iii) row space, $\mathbf{R}(A')$

(iv) left null space, $\mathbf{N}(A')$

Every matrix transforms its row space to its column space.

[1] In this appendix we focus on quadratic matrices only.

Matrix operations We define the following matrix operations:

(i) addition: $C = A + B$ is the matrix with the ij-th element equal to $c_{ij} = a_{ij} + b_{ij}$.

(ii) multiplication: $C = AB$ is the matrix with the ij-th element equal to $c_{ij} = \sum_{k=1}^{d} a_{ik}b_{kj}$. Note that the multiplication is in general not commutative, i.e. $AB \neq BA$.

(iii) transposition: A' is the matrix with the indices i and j interchanged, i.e. the ij-th element is a_{ji}. Note that $(A')' = A$

The matrix addition and multiplication are associative and distributive. The *rank of a matrix* A denoted by $\text{rank}(A)$ is the maximal number of linearly independent columns or rows. The following relations hold: $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$, $\text{rank}(A') = \text{rank}(A)$, and $\text{rank}(A) = \text{rank}(AA') = \text{rank}(A'A)$. The *index of a matrix* is the smallest integer k such $\text{rank}A^k = \text{rank}A^{k+1}$. For nonsingular matrices $\text{index}A = 0$. Alternative characterizations of the index of a matrix can be found in Meyer (2000, p.395).

Special matrices Special type of matrices are:

(i) zero matrix 0_d: $a_{ij} = 0$ for all $i, j = 1, \ldots, d$. Most of the time the index d is suppressed.

(ii) identity matrix I_d: $a_{ii} = 1$ for all $i = 1, \ldots, d$ and $a_{ij} = 0$ for all $i, j = 1, \ldots, d$ and $i \neq j$. I_d has ones on the diagonal and zeros elsewhere.

(iii) diagonal matrix: all off-diagonal elements a_{ij}, $i \neq j$, are equal to zero.

(iv) lower triangle matrix: all elements a_{ij}, $i < j$, are equal to zero.

(v) upper triangle matrix: all elements a_{ij}, $i > j$, are equal to zero.

(vi) symmetric matrix: $a_{ij} = a_{ji}$ or $A' = A$.

(vii) skew symmetric matrix: $a_{ij} = -a_{ji}$ or $A' = -A$. Hence, skew symmetric matrices have zeros on the diagonal.

(viii) inverse matrix: a matrix A^{-1} is called the inverse of A if $AA^{-1} = A^{-1}A = I_d$. A matrix with no inverse is called *singular*. Note that $(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$, and $(A')^{-1} = (A^{-1})'$. An invertible matrix has necessarily rank d.
The set of all invertible $d \times d$ matrices, also known as the general linear group, is denoted by $\mathbb{GL}(d)$.

(ix) orthogonal matrix: $A'A = AA' = I_d$. Hence, the transpose of A is its inverse.

(x) nilpotent matrix: there exits an integer $k > 0$ such that $A^k = 0$. The smallest such number is called the index of A.

(xi) projector or idempotent matrix: $A^2 = A$.

Matrix functions Matrix functions are functionals from $\mathbb{M}(d) \to \mathbb{R}$. The two most popular matrix functions are the trace, $\mathrm{tr}(A)$, and the determinant, $\det(A)$ or $|A|$.

The trace of A is defined as the sum of its diagonal elements, i.e. $\mathrm{tr}(A) = \sum_{i=1}^{d} a_{ii}$. Some properties of the trace are:

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$
$$\mathrm{tr}(ABC) = \mathrm{tr}(BAC) = \mathrm{tr}(CAB)$$

The determinant is characterized by the following three properties:

(i) $\det I_d = 1$

(ii) The determinant changes sign when two rows are interchanged.

(iii) The determinant is a linear function of each row taken separately.

These three properties imply the following rules:

(i) If two rows are equal $\det A = 0$.

(ii) Subtracting a multiple of one row from another row leaves $\det A$ unchanged.

(iii) A matrix with a row of zeros has $\det A = 0$.

(iv) If A is triangular or diagonal, $\det A = a_{11}a_{22}\ldots a_{dd}$.

(v) $\det A = 0 \Leftrightarrow A$ is singular. $\det A \neq 0 \Leftrightarrow A$ is invertible.

(vi) $\det(AB) = \det A \det B$.
Hence, $\det AB = \det BA$ and $\det A^{-1} = (\det A)^{-1}$.

(vii) $\det A = \det A'$. Every rule for rows is true for columns.

(viii) For any orthogonal matrix Q, $\det Q = \pm 1$.
Proof: $1 = \det I_n = \det Q'Q = \det Q' \det Q = (\det Q)^2$.

Matrix Norm A matrix norm is a norm on the vector space $\mathbb{M}(d)$, the set of all $d \times d$ matrices. Thus, for $A \in \mathbb{M}(d)$, $\|A\| : \mathbb{M}(d) \to \mathbb{R}$ must satisfy the usual requirements:

(i) $\|A\| \geq 0$, and $\|A\| = 0$ if and only if $A = 0$.

(ii) $\|\alpha A\| = |\alpha| \, \|A\|$ for all $\alpha \in \mathbb{R}$.

(iii) $\|A + B\| \leq \|A\| + \|B\|$ for all $A, B \in \mathbb{M}(d)$.

In addition, we require for the matrix norm to be *sub-multiplicative*:

$$\|AB\| \leq \|A\|\|B\| \quad \text{for all } A, B \in \mathbb{M}(d).$$

For a detailed introduction to matrix norms see Meyer (2000, section 5.2).

A matrix norm can be specified as being induced from any vector norm on $\mathbb{R}^d$. If $\|x\|$ is any vector norm on $\mathbb{R}^d$ then we may consider the *induced matrix norm* or *operator norm*:

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

Thus the induced matrix norm is the maximum amount a vector on the unit sphere can be stretched.[2] The induced matrix norm is sub-multiplicative which implies $\|Ax\| \leq \|A\|\|x\|$. Moreover, Gelfand's formula holds:

$$\lim_{n\to\infty} \|A^n\|^{1/n} = \rho(A)$$

where $\rho(A)$ denotes the spectral radius of A.

Some common induced matrix norms are:

(i) If $\|x\|_1 = \sum_{j=1}^d |x_j|$, $\|A\|_1 = \max_{1\leq j\leq d} \sum_{i=1}^d |a_{ij}|$, the maximum absolute column sum.

(ii) If $\|x\|_2 = \sqrt{x_1^2 + \ldots + x_d^2}$, the Euclidean norm, $\|A\|_2 = \sqrt{\rho(A'A)}$ which equals the largest singular value of A.

(iii) If $\|x\|_\infty = \max_{1\leq j\leq d}\{|x_j|\}$, $\|A\|_\infty = \max_{1\leq i\leq d} \sum_{j=1}^d |a_{ij}|$, the maximum absolute row sum.

[2]Because the norm is a continuous function and because the unit sphere is compact, the maximum is attained.

Another way to introduce matrix norms is to view A as an element of the vector space $\mathbb{R}^{d^2}$. Using the Euclidean norm this leads to the Frobenius or Hilbert-Schmidt matrix norm:

$$\|A\| = \sqrt{\sum_{i,j}^{n} a_{ij}^2} = \sqrt{\mathrm{tr}(A'A)} = \sqrt{\sum_{j=1}^{d} \lambda_i}$$

where λ_j are the eigenvalues of $A'A$. Thus the Frobenius norm stakes the columns of A into a long d^2-dimensional vector and takes its Euclidian norm.

Because all norms are equivalent in finite dimensional spaces, i.e. for two norm $\|.\|_\alpha$ and $\|.\|_\beta$ there exists positive scalars a and b such that $a\|A\|_\alpha \leq \|A\|_\beta \leq b\|A\|_\alpha$, it does not really matter which one we will use. The choice can therefore be made on the basis of convenience.

Generalized Inverse Let A be a singular matrix of index k with rank $A^k = r$. Then there exists a nonsingular matrix Q such that

$$Q^{-1}AQ = \begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix}$$

where C is a $r \times r$ nonsingular matrix and N is a nilpotent matrix of index k (see Meyer, 2000, pp.397). This block diagonal matrix is called a *core-nilpotent decomposition* of A. When A is nonsingular, $k = 0$ and $r = d$, so N is not present and $Q = I_d$ and $C = A$.

The core-nilpotent decomposition allows the definition of the *Drazin inverse* denoted by A^D:

$$A^D = Q \begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}.$$

This is an example of a generalized inverse. It is unique and characterized algebraically by the following three properties:

(i) $A^D A A^D = A^D$

(ii) $AA^D = A^D A$

(iii) $A^{k+1}A^D = A^k$

If index$A \leq 1$, the Drazin inverse is called the *group inverse*. It has the additional property that $AA^D A = A$.

The Drazin inverse has to be distinguished from the Moore-Penrose generalized inverse $A^\dagger$. The latter is uniquely characterized by

(i) $AA^\dagger A = A$

(ii) $A^\dagger AA^\dagger = A^\dagger$

(iii) $(AA^\dagger)' = AA^\dagger$

(iv) $(A^\dagger A)' = A^\dagger A$

If A is an $n \times m$, $n > m$, with m linearly independent columns, the Moore-Penrose inverse becomes the least-squares inverse $A^\dagger = (A'A)^{-1}A'$. For further details on generalized inverses consults Campbell and Meyer (1979).

C.3 Eigenvalues and Eigenvectors

Given a $d \times d$ matrix A, the scalars λ and vectors $x \neq 0$ satisfying $Ax = \lambda x$ are called *eigenvalues* and *eigenvectors* of A and any pair (λ, x) is called an *eigenpair* of A. The set of all distinct eigenvalues is denoted by $\sigma(A)$ and is called the *spectrum* of A. These definitions imply

- $\lambda \in \sigma(A) \Leftrightarrow A - \lambda I_d$ is singular $\Leftrightarrow \det(A - \lambda I_d) = 0$.
- The null space of $A - \lambda I_d$, denoted by $\mathbf{N}(A - \lambda I_d)$, is the set of all eigenvectors associated with λ and is called an *eigenspace*. $\mathbf{N}(A - \lambda I_d)$ is a subspace of $\mathbb{R}^d$. The dimension of $\mathbf{N}(A - \lambda I_d)$ is called the *geometric multiplicity* of λ. Thus, the geometric multiplicity gives the maximal number of linearly independent eigenvectors associated to a given eigenvalue.
- The largest modulus of all eigenvalue is called the *spectral radius* of A and is denoted by $\rho(A)$, i.e. $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$.

The characterization of the eigenvalues as scalars satisfying $\det(A - \lambda I_d) = 0$ implies that the eigenvalues are the roots (zeros) of the polynomial $\mathcal{P}(\lambda) = \det(A - \lambda I_d)$. As $\mathcal{P}(\lambda)$ is a polynomial of degree d, called the *characteristic polynomial*, all eigenvalues must satisfy the *characteristic equation* $\mathcal{P}(\lambda) = 0$. As the leading term of the characteristic polynomial is $(-1)^d\lambda^d$ and thus nonzero, there are d eigenvalues. Some of these eigenvalues may, however, appear several times. In this case we speak of *repeated eigenvalues*. The number of times an eigenvalue is repeated is called the *algebraic multiplicity* of λ. If λ appears only once the eigenvalue is called *simple*. Although A contains just real entries some of the eigenvalues may be complex numbers. In this case, the eigenvalues appear as conjugate pairs, i.e. if $\lambda \in \sigma(A)$ then $\bar{\lambda} \in \sigma(A)$, and the eigenvectors will have complex entries to ensure that

$Ax = \lambda x \in \mathbb{R}^d$. It is clear that eigenvectors are not uniquely determined: if x is an eigenvector, αx, $\alpha \in \mathbb{R}$, is also an eigenvector corresponding to the same eigenvalue. In numerical applications eigenvectors are therefore normalized: f.e. its first entry is set to one or the length of x, $\|x\|$, is set to one.

The eigenvalues depend *continuously* on the entries of the matrix A. This is, however, not true for the eigenvectors.

From the fundamental theorem of algebra we know that the characteristic polynomial can be factorized as

$$\mathcal{P}(\lambda) = \det(A - \lambda I_d) = (\lambda_1 - \lambda) \dots (\lambda_d - \lambda)$$

where that λ_i's denote the eigenvalues. Setting $\lambda = 0$, we see that $\det(A) = \lambda_1 \dots \lambda_d$, i.e. the determinant of A equals the product of all eigenvalues. In addition, one can prove that the sum of the eigenvalues equals the trace of A. Hence, $\mathrm{tr}A = \lambda_1 + \ldots + \lambda_n$. The *index of an eigenvalue* λ is defined as the index of the matrix $(A - \lambda I_d)$, i.e. the smallest integer k such that $\mathrm{rank}(A - \lambda I_d)^k = \mathrm{rank}(A - \lambda I_d)^{k+1}$.

C.4 Similarity and Jordan Form

Two matrices A and B are called *similar* if there exists a nonsingular matrix Q such that $A = QBQ^{-1}$. Obviously, two similar matrices share the same eigenvalues. An important special case arises if B is a diagonal matrix. The matrix A is then called *diagonalizable*. For a diagonalizable matrix, $AQ = Q\Lambda$ where Λ is a diagonal matrix which holds the eigenvalues of A on the diagonal. The eigenvectors of A are then the columns of Q. On the other hand, if A has d linearly independent eigenvectors then A is diagonalizable where the columns of Q are made up by the d eigenvectors. Note that the diagonalizing matrix is not unique. The equation $AQ = Q\Lambda$ only holds if the columns of Q are the eigenvectors of A. Other matrices Q will not produce a diagonal Λ.

A matrix with distinct eigenvalues can be diagonalized. Its eigenvectors are not necessarily orthogonal to each other. *Normal* matrices are matrices which commute with their transpose, i.e. $AA' = A'A$.[3] These matrices are exactly those which are orthogonally similar to a diagonal matrix, i.e. there exists an orthogonal matrix Q such that $A = Q\Lambda Q'$. Examples of such real matrices include symmetric matrices ($A = A'$), skew-symmetric ($A = -A'$), and orthogonal matrices ($A'A = AA' = I_d$). A is symmetric if and only if A

[3] We just consider matrices with real entries.

is orthogonally similar to a real diagonal matrix. Skew symmetric matrices have pure imaginary eigenvalues.

The following theorem is important for the geometric interpretation of linear difference equations.

Theorem C.1 (Spectral Decomposition)**.** A matrix A with spectrum $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ is diagonalizable if and only if there exist matrices $\{P_1, P_2, \ldots, P_s\}$ such that

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_s P_s \tag{C.1}$$

where the P_j's have the following properties:

(i) Each P_j is a projection onto $\mathbf{N}(A - \lambda_j I_d)$ along $\mathbf{R}(A - \lambda_j I_d)$;

(ii) $P_i P_j = 0$ for $i \neq j$;

(iii) $P_1 + P_2 + \ldots + P_s = I_d$.

The expansion (C.1) is known as the spectral decomposition of A, and the P_j's are called the spectral projectors associated with A.

An implication of the Spectral Decomposition (C.1) is a corresponding decomposition of A^t.

Corollary C.1. If A has Spectral Decomposition (C.1), A^t has Spectral Decomposition

$$A^t = \lambda_1^t P_1 + \lambda_2^t P_2 + \ldots + \lambda_s^t P_s.$$

The spectral projectors take a particularly simple form if the corresponding eigenvalue is simple.

Corollary C.2. If $A = Q\Lambda Q^{-1}$ with a simple eigenvalue λ, then the corresponding spectral projector P_λ is

$$P_\lambda = \frac{q_\lambda q^\lambda}{q^\lambda q_\lambda}$$

where q_λ and q^λ are right and left eigenvectors associated with λ. Thus, q_λ and q^λ are given by the corresponding column in Q, respectively row in Q^{-1}.

Unfortunately, not all matrices are diagonalizable. These, so-called *defective* matrices, do not have d linearly independent eigenvectors. From the arguments presented above it is clear that a failure of diagonalizability can only arise if some eigenvalues are repeated. The reverse is, however, not true. Even if some eigenvalues are repeated there can still exist a full set of linearly

independent eigenvectors. Take, f.e. the identity matrix as an example: it has one eigenvalue 1 which is repeated d times and nevertheless has a full set of d linearly independent eigenvectors. The eigenvalues is this case are called *semisimple*. For these eigenvectors the algebraic multiplicity equals its geometric multiplicity. If the matrix has a deficiency of eigenvectors it cannot be diagonalizable, but it can be transformed by similarity to a *block diagonal form* known as the *Jordan form*. This makes the matrix as close to a diagonal matrix as possible.

Theorem C.2 (Jordan Form)**.** Let A be a $d \times d$ matrix with distinct eigenvalues $\sigma(A) = \{\lambda_1, \dots \lambda_s\}$. Then there exists is nonsingular matrix Q such that

$$Q^{-1}AQ = J = \begin{pmatrix} J(\lambda_1) & 0 & \cdots & 0 \\ 0 & J(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J(\lambda_s) \end{pmatrix} \tag{C.2}$$

where J consists of s *Jordan segments* $J(\lambda_j)$, one for each eigenvalue $\lambda_j \in \sigma(A)$. Each segment $J(\lambda_j)$ is made up of $m_j = \dim \mathbf{N}(A - \lambda_j I_d)$ *Jordan blocks* $J_l(\lambda_j)$, $l = 1, \dots, m_j$:

$$J(\lambda_j) = \begin{pmatrix} J_1(\lambda_j) & 0 & \cdots & 0 \\ 0 & J_2(\lambda_j) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{m_j}(\lambda_j) \end{pmatrix} \text{ with } J_l(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}.$$

The largest Jordan block in $J(\lambda_j)$ is $k_j \times k_j$ where $k_j = \text{index}(\lambda_j)$. The number of $i \times i$ Jordan blocks in $J(\lambda_j)$ is given by

$$\nu_i(\lambda_j) = r_{i-1}(\lambda_j) - 2r_i(\lambda_j) + r_{i+1}(\lambda_j)$$

where $r_i(\lambda_j) = \text{rank}(A - \lambda_j I_d)^i$. Note that Jordan blocks $J_l(\lambda)$ can be written as $J_l(\lambda) = \lambda I_i + N_i$ where i is the size of the block and N_i is a nilpotent matrix of index $i - 1$, i.e. $N^{i-1} = 0$.

The structure of the Jordan form is unique in the sense the number of Jordan segments as well as the number and sizes of Jordan blocks in each segment is uniquely determined by A. Every matrix similar to A has the same Jordan form. Clearly, the matrix Q is not unique.

In order to clarify these ideas, consider the following example from Meyer

(2000, p.590-591):

$$A = \begin{pmatrix} 5 & 4 & 0 & 0 & 4 & 3 \\ 2 & 3 & 1 & 0 & 5 & 1 \\ 0 & -1 & 2 & 0 & 2 & 0 \\ -8 & -8 & -1 & 2 & -12 & -7 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -8 & -8 & -1 & 0 & -9 & -5 \end{pmatrix}.$$

The characteristic polynomial is $\mathcal{P}(\lambda) = \lambda^6 - 6\lambda^5 + 9\lambda^4 + 8\lambda^3 - 24\lambda^2 + 16 = (\lambda - 2)^4(\lambda + 1)^2$. Hence, there are two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$ with algebraic multiplicity 4 for and 2, respectively. This implies that there are two Jordan segments $J(2)$ and $J(-1)$. The next step is the computation of the ranks $r_i(2) = \text{rank}((A - 2I_6)^i)$ and $r_i(-1) = \text{rank}((A + I_6)^i)$ until $r_k(\lambda_j) = r_{k+1}(\lambda_j)$:

$$\begin{aligned} r_1(2) &= \text{rank}(A - 2I_6) \quad = 4 & \qquad r_1(-1) &= \text{rank}(A + I_6) = 4 \\ r_2(2) &= \text{rank}((A - 2I_6)^2) = 3 & \qquad r_2(-1) &= \text{rank}(A + I_6) = 4 \\ r_3(2) &= \text{rank}((A - 2I_6)^3) = 2 \\ r_4(2) &= \text{rank}((A - 2I_6)^4) = 2 \end{aligned}$$

so that the index of $\lambda_1 = 2$ and $\lambda_2 = -1$ are $k_1 = 3$ and $k_2 = 1$, respectively. Hence, the largest Jordan block of $J(2)$ is 3×3 while the largest Jordan block of $J(-1)$ is 1×1. Thus, the eigenvalue $\lambda_2 = -1$ is semisimple and the Jordan segment $J(-1)$ is a diagonal matrix. To determine the number of $i \times i$ Jordan blocks in $J(2)$, we compute

$$\begin{aligned} \nu_3(2) = r_2(2) - 2r_3(2) + r_4(2) = 1 \quad &\Rightarrow \quad \text{one } 3 \times 3 \text{ block in } J(2) \\ \nu_2(2) = r_1(2) - 2r_2(2) + r_3(2) = 0 \quad &\Rightarrow \quad \text{no } 2 \times 2 \text{ block in } J(2) \\ \nu_1(2) = r_0(2) - 2r_1(2) + r_2(2) = 1 \quad &\Rightarrow \quad \text{one } 1 \times 1 \text{ block in } J(2) \\ \nu_1(-1) = r_0(2) - 2r_1(2) + r_2(2) = 2 \quad &\Rightarrow \quad \text{two } 1 \times 1 \text{ blocks in } J(-1). \end{aligned}$$

This implies that

$$J(2) = \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right) \quad \text{and} \quad J(-1) = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & -1 \end{array}\right).$$

so that finally

$$J = \begin{pmatrix} J(2) & 0 \\ 0 & J(-1) \end{pmatrix}.$$

To complete the computation of the Jordan form, the structure of the similarity matrix Q must be determined. For this purpose it is sufficient to focus on a single $i \times i$ Jordan block $J_l(\lambda)$ in the Jordan segment $J(\lambda)$. Denote by $Q_l = (x_1, x_2, \ldots, x_i)$ the portion of $Q = (\ldots | Q_l | \ldots)$ that corresponds to the position of $J_l(\lambda)$ in J. Because $AQ = QJ$, implies $AQ_l = Q_l J_l(\lambda)$, we arrive at the following equation system:

$$A(x_1, x_2, \ldots, x_i) = (x_1, x_2, \ldots, x_i) \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}_{i \times i},$$

or more explicitly

$$\begin{array}{lcllcll} Ax_1 = \lambda x_1 & \Rightarrow & x_1 \text{ is an eigenvector} & \Rightarrow & (A - \lambda I_d)x_1 & = 0 \\ Ax_2 = x_1 + \lambda x_2 & \Rightarrow & (A - \lambda I_d)x_2 = x_1 & \Rightarrow & (A - \lambda I_d)^2 x_2 & = 0 \\ \vdots & & \vdots & & \vdots & \\ Ax_i = x_{i-1} + \lambda x_i & \Rightarrow & (A - \lambda I_d)x_i = x_{-1} & \Rightarrow & (A - \lambda I_d)^i x_i & = 0. \end{array}$$

Thus, x_1, the first column of Q_l, is an eigenvector. The other vectors are called *generalized eigenvectors* or more preciesely *generalized eigenvectors of order k* if $x \in \mathbf{N}((A - \lambda I_d)^k)$, but $x \notin \mathbf{N}((A - \lambda I_d)^{k-1})$.[4]

Continuing with the numerical example from above, we first take the 3×3 Jordan block $J_1(2)$. A corresponding eigenvector is $x_1 = (1, 0, -1, -1, 0, -1)'$. A generalized eigenvector of order one for the eigenvalue $\lambda 02$ must then fulfill $(A - 2I_6)x_2 = x_1$ or $(A - 2I_6)^2 x_2 = 0$. Such vector is given by $x_2 = (0, 1, 0, -1, 0, -1)'$. A generalized eigenvector of order two then must satisfy $(A - 2I_6)x_3 = x_2$ or $(A - 2I_6)^3 x_3 = 0$. Such a vector is given by $x_3 = (1, 0, 0, -1, 0, -1)'$. Hence, $Q_l = (x_1, x_2, x_3)$. A second linearly independent eigenvector for the eigenvalue $\lambda = 2$ is $x_4 = (-1, 0, 1, -8, 0, 1)'$. For the eigenvalue $\lambda = -1$ a full set of eigenvectors exists. Two linearly independent eigenvectors are $x_5 = (1, 0, 0, -2, 0, -2)'$ and $x_6 = (11, 2, 2, -24, -2, -22)'$. Putting these eigenvectors together gives the similarity matrix Q:

$$Q = \begin{pmatrix} 1 & 0 & 1 & -1 & 1 & 11 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 1 & 0 & 2 \\ -1 & -1 & -1 & -8 & -2 & -24 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ -1 & -1 & -1 & 1 & -2 & -22 \end{pmatrix}.$$

[4] For more details see Meyer (2000, pp.593).

Although the matrix A consists of real numbers only, its Jordan form can still have complex entries because the roots of the characteristic polynomial are not necessarily real. However, they appear as conjugate pairs so that it is possible to construct a *real block diagonal Jordan form* with only real Jordan blocks (see Colonius and Kliemann, 2014, section 1.2). The Jordan blocks corresponding to complex eigenvalues consist of blocks of 2×2 matrices. Suppose that $\lambda = \alpha + \imath\beta$, $\beta \neq 0$, is a complex eigenvalue then its conjugate $\bar{\lambda} = \alpha - \imath\beta$ is also an eigenvalue. By observing that

$$C(\lambda) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = Q \begin{pmatrix} \alpha + \imath\beta & 0 \\ 0 & \alpha - \imath\beta \end{pmatrix} Q^{-1}$$

with

$$Q = \begin{pmatrix} -\imath & 1 \\ -1 & \imath \end{pmatrix} \qquad \text{and} \qquad Q^{-1} = \frac{1}{2} \begin{pmatrix} \imath & -1 \\ 1 & -\imath \end{pmatrix},$$

the real Jordan form $J^{\mathbb{R}}$ can be constructed as follows. For the real eigenvalues one proceeds as before and for the complex eigenvalues $\lambda = \alpha \pm \imath\beta$, $\beta \neq 0$, the Jordan blocks are

$$J_l(\lambda) = \begin{pmatrix} C(\lambda) & I_2 & \cdots & \cdots & 0 \\ 0 & C(\lambda) & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & C(\lambda) & I_2 \\ 0 & \cdots & \cdots & \cdots & C(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & -\beta & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \beta & \alpha & 0 & 1 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \alpha & -\beta & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \beta & \alpha & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \alpha & -\beta & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \beta & \alpha & 0 & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \alpha & -\beta \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \beta & \alpha \end{pmatrix}.$$

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