Introduction to Dynamic Programming Lecture Notes*

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Notation and Symbols

- \mathcal{X} state space
- U control
- μ strategy
- π policy
- V value function

1 Introduction

The foundation of macroeconomic theory on microeconomic principles has been one of the most important developments in economics. By now it is standard to view the decision maker (households, firms, state) as operating in a complex stochastic environment. In particular, agents are conceived as players in a dynamic stochastic game. Typically, it is assumed that the players understand the rules of the game and can foresee the consequences of their actions on themselves and on others. The agents are thus understood to choose *strategies* which maximize their objective function.

From a modeling point of view, it is therefore necessary to specify exactly the information at the disposal of agents, the technology available to them and the restrictions which constrain their actions. The decisions depend typically on expectations about the future. These expectations will influence the actions of the agents already today, thereby determining the possibilities of the agent in the future. This intertemporal interaction is visualized in figure 1. It is also important to emphasize that this strategic intertemporal interaction is typical for economics system and differentiate them from physical systems.

2 Basic concepts

In order to understand the issues involved in Dynamic Programming, it is instructive to start with the simple example of inventory management. Denote the stock of inventory at the beginning of period t by X_t , then the manager has to decide on how much to order to replenish the stock. The order U_t is considered to be the *control variable*. In each period the inventory is reduced by satisfying a stochastic demand Z_t . It is assumed that the manager does not know the realized value of demand at the time he makes the decision. The situation is depicted in figure 2.

In this problem the variable which characterizes the state of the inventory, in our case X_t , is called the state variable of the system. The state variable or shortly the state must lie in some set called the state space denoted by



Figure 1: Intertemporal Macroeconomics



Figure 2: Inventory management

 \mathcal{X} . The control variable or control, for short, takes values in some set \mathcal{C} . As the demand Z_t is assumed to be identically and independently distributed, the state is just the inventory carried over from last period. Because next period's inventory is just given by the accounting identity

$$X_{t+1} = X_t + U_t - Z_t, (2.1)$$

the control could as well be X_{t+1} . In a more general context, demand follows a stochastic process $\{Z_t\}$ governed by some transition function Q where tomorrow's demand depends on the demand realized today and possibly on the control U_t . In this more general setup, the state is given by (X_t, Z_t) .

In each period, the manager faces some costs. In our example the costs are twofold. On the one hand inventory holdings are costly. This cost is denoted by $h(X_{t+1})$. The inventory costs may also account for shortage cost for unfilled order if $X_{t+1} < 0$. On the other hand, each order produces some cost $c U_t$. In each period total costs amount to:

$$c U_t + h(X_{t+1})$$
 (2.2)

The transition equation (2.1) and the cost or utility function (period valuation) are the main ingredients of the inventory problem. The objective of the manager is to minimize expected discounted costs:

$$J(x_0) = \mathbb{E}_0 \sum_{t=0}^{T-1} \beta^t (c \ U_t + h(X_{t+1}))$$

= $\mathbb{E}_0 \sum_{t=0}^{T-1} g_t(X_t, U_t, Z_t) + g_T(X_T) \longrightarrow \min_{U_t}, \qquad 0 < \beta < 1 \qquad (2.3)$

starting in period 0 with inventory x_0 . The optimization is subject to a *feasibility constraint*

$$U_t \in \Gamma(X_t) = [0, B - X_t] \subseteq \mathcal{C} = [0, B]$$
(2.4)

where B is the maximal storage capacity.

It is clear that it is not optimal to set the controls U_0, \ldots, U_{T-1} in advance at time 0 without knowing the realizations of demand. It is thus definitely better to decide upon the order at time t after knowing the state X_t . We are therefore confronted with a sequential decision problem. The gathering of information, here observing the state X_t , becomes essential. This way of viewing the decision problem implies that we are actually not interested in setting numerical values for U_t in each period, but in a strategy, rule, reaction function, or policy function μ_t which assigns to each possible state X_t an action $\mu_t(X_t) = U_t$.¹

The control U_t must lie in some subset $\Gamma(X_t) \subseteq \mathcal{C}$, the control constraint set at X_t . Γ assigns to every state X_t a set $\Gamma(X_t) = [0, B - X_t]$ and is thus a set valued function or correspondence from \mathcal{X} into the subsets of \mathcal{C} . As noted in the footnote before, in a general stochastic context Γ may depend on X_t and Z_t . It is typically assumed that $\Gamma(x) \neq \emptyset$ for all $x \in \mathcal{X}$. This implies that there is always a feasible choice to make.

Let $M = \{\mu : \mathcal{X} \to \mathcal{C} \text{ s.t. } \mu(x) \in \Gamma(x)\}$. We call $\pi = (\mu_0, \mu_1, \dots, \mu_{T-1})$ a feasible policy if $\mu_t \in M$. The set of all feasible policies is called the *policy* space and is denoted by Π . If $\pi = (\mu, \mu, \dots, \mu)$, π is called a stationary policy. In our example $\mathcal{X} = [0, B], \mathcal{C} = [0, B], \text{ and } \Gamma(X_t) = [0, B - X_t].$

With this notation we can rewrite our decision problem as

$$J_{\pi}(X_0) = \mathbb{E}_0 \sum_{t=0}^{T-1} g_t(X_t, \mu_t(X_t), Z_t) + g_T(X_T) \longrightarrow \min_{\pi \in \Pi}$$
(2.5)

where $J_{\pi}(X_0)$ is called the value function at X_0 . The expectation is taken with respect to the probability measure on $(Z_0, Z_1, ...)$ induced by Q given initial conditions (x_0, z_0) . An optimal policy is thus a policy π^* which minimizes the cost functional given the initial conditions x_0 . The optimal value function or optimal cost function $J^*(x_0)$ is defined as

$$J^*(x_0) = \inf_{\pi \in \Pi} J_{\pi}(x_0)$$
(2.6)

The optimal policy thus satisfies: $J^*(x_0) = J_{\pi^*}(x_0)$. It is important to realize the sequential nature of the problem which makes the gathering of information about the state of the system indispensable:

¹In case $\{Z_t\}$ is not i.i.d. the policy function can also depend on Z_t .

- 1. The decision maker observes the state of the system (i.e. X_t) and applies his decision rule $U_t = \mu_t(X_t)$.
- 2. The demand Z_t is realized according to some transition function Q.
- 3. Cost or benefit $g_t(X_t, \mu_t(X_t), Z_t)$ is incurred and added to the costs or benefits of previous periods. In the last period cost or benefit is $g_T(X_T)$ only.
- 4. The state in the next period is constructed according to the transition function $X_{t+1} = f_t(X_t, \mu_t(X_t), Z_t) = X_t + U_t Z_t$.
- 5. Start again in 1.

3 The Bellman equation and the principle of optimality

The recursive nature of the decision problem leads to the *principle of opti*mality due to Bellman. This principle is at the heart of the dynamic programming technique and is intimately related to the idea of time consistency (see Kydland and Prescott, 1977). Suppose we have selected an optimal policy $\pi^* = (\mu_0^*, \mu_1^*, \ldots, \mu_{T-1}^*)$. Consider then the subproblem arising in period $\tau, 0 < \tau < T - 1$, when the system is in state X_{τ} :

$$\mathbb{E}_{\tau} \sum_{t=\tau}^{T-1} g_t(X_t, \mu_t(X_t), Z_t) + g_T(X_T) \longrightarrow \min A$$

The principle of optimality then states that the truncated policy $(\mu_{\tau}^*, \mu_{\tau+1}^*, \dots, \mu_{T-1}^*)$ is also optimal for the above subproblem. The intuition is simple: if the truncated policy were not optimal for the subproblem, the decision maker would be able to reduce cost further by switching to an optimal policy for the subproblem once X_{τ} has been reached. This idea can be exploited by solving the decision problem by *backward induction*. For this purpose consider the decision problem in period T-1 with given state X_{T-1} . Clearly the decision maker chooses $U_{T-1}^* = \mu_{T-1}(X_{T-1}) \in \Gamma(X_{T-1})$ in order to minimize

$$\mathbb{E}_{T-1}\left[g_{T-1}(X_{T-1},\mu_{T-1}(X_{T-1}),Z_{T-1})+g_T(X_T)\right].$$

Denote the optimal cost for the last period by $J_{T-1}(X_{T-1})$:

$$J_{T-1}(X_{T-1}) = \min_{U_{T-1} \in \Gamma(X_{T-1})} \mathbb{E}_{T-1}[g_{T-1}(X_{T-1}, \mu_{T-1}(X_{T-1}), Z_{T-1}) + g_T(\underbrace{f_{T-1}(X_{T-1}, U_{T-1}, Z_{T-1})}_{=X_T})]$$

 J_{T-1} is clearly a function of X_{T-1} . In the process of computing $J_{T-1}(X_{T-1})$, we automatically obtain the optimal policy $U_{T-1}^* = \mu_{T-1}^*(X_{T-1}) \in \Gamma(X_{T-1})$ for period T-1. Going back one period to period T-2 with state X_{T-2} , the decision maker should not just minimize expected period cost but take the consequence of his decision for period T-1 into account. Thus he should minimize the expected period cost in T-2 plus the expected period in T-1given that an optimal policy is followed in period T-1. He must therefore minimize:

$$\mathbb{E}_{T-2}\left[g_{T-2}(X_{T-2},\mu_{T-2}(X_{T-2}),Z_{T-2}) + J_{T-1}(X_{T-1})\right] \longrightarrow \min_{U_{T-2}=\mu_{T-2}(X_{T-2})\in\Gamma(X_{T-2})}$$

Denoting the optimal cost by $J_{T-2}(X_{T-2})$, we get:

$$J_{T-2}(X_{T-2}) = \min_{U_{T-2} = \mu_{T-2}(X_{T-2}) \in \Gamma(X_{T-2})} \mathbb{E}_{T-2}[g_{T-2}(X_{T-2}, \mu_{T-2}(X_{T-2}), Z_{T-2}) + J_{T-1}(X_{T-1})]$$

Going back further in time we obtain

$$J_t(X_t) = \min_{U_t = \mu_t(X_t) \in \Gamma(X_t)} \mathbb{E}_t[g_t(X_t, \mu_t(X_t), Z_t) + J_{t+1}(X_{t+1})]$$

where X_{t+1} can be substituted by $X_{t+1} = f_t(X_t, \mu_t(X_t), Z_t)$. From this reasoning we can derive the following proposition.

Proposition 3.1. If $J^*(X_0)$ is the optimal cost, $J^*(X_0) = J_0(X_0)$. Moreover,

$$J_t(X_t) = \min_{U_t \in \Gamma(X_t)} \mathbb{E}_t[g_t(X_t, U_t, Z_t) + J_{t+1}(f_t(X_t, U_t, Z_t))], \quad t = 0, 1, \dots, T-1,$$

and $J_T(X_T) = g_T(X_T)$. Furthermore, if $U_t^* = \mu_t^*(X_t) \in \Gamma(X_t)$ minimizes the right hand side above for each X_t , then the policy $\pi^* = (\mu_0^*, \mu_1^*, \dots, \mu_{T-1}^*)$ is optimal.

In macroeconomics, we have, typically, a maximization instead of a minimization problem and the following time invariant specification which leads us to consider only time invariant policies $\pi = (\mu, \mu, \dots, \mu)$:

- $g_t(X_t, \mu(X_t), Z_t) = \beta^t U(X_t, \mu(X_t));$
- $X_{t+1} = f(X_t, \mu(X_t), Z_{t+1});$
- $J_T(X_T) = 0.$

The above iteration can therefore be rewritten as follows:

$$J_{t}(X_{t}) = \max_{U_{t}=\mu(X_{t})\in\Gamma(X_{t})} \mathbb{E}_{t}[\beta^{t}U(X_{t},\mu(X_{t})) + J_{t+1}(f(X_{t},\mu(X_{t}),Z_{t}))]$$

$$\implies$$
$$V_{t+1}(X_{t}) = \max_{U_{t}=\mu(X_{t})\in\Gamma(X_{t})} U(X_{t},\mu(X_{t})) + \beta\mathbb{E}_{t}V_{t}(f(X_{t},\mu(X_{t}),Z_{t}))$$

where $V_{T-t}(x) = \beta^{-t} J_t(x)$.

We might then consider the limit

$$\lim_{t \to \infty} V_t(x) = V(x).$$

If this limit exists, it must satisfy the following *functional equation*, called *Bellman equation*:

$$V(x) = \max_{u \in \Gamma(x)} U(x, u) + \beta \mathbb{E} V(x')$$
(3.1)

$$= \max_{u \in \Gamma(x)} U(x, u) + \beta \mathbb{E} V(f(x, u, z'))$$
(3.2)

where a prime denotes next period's value. The expectation \mathbb{E} is conditional on the information available to the agent. It must be emphasized that in the Bellman equation the unknown is not a number as in standard algebraic equations, but a function. In this context the following mathematical issues arise:

- 1. Does the limit exist? Is the limit independent from the initial functional V_0 ?
- 2. Does the Bellman equation have a unique solution? Can it be found by iteration irrespective of the starting functional?
- 3. Does there exist a time invariant policy function μ ? What are the properties of such a function?
- 4. Is V and/or μ differentiable? If so we obtain an analogue to the envelope theorem, the so-called *Benveniste-Scheinkman formula*:

$$\frac{\partial V(x)}{\partial x} = \frac{\partial U(x,u)}{\partial x} + \beta \mathbb{E} \frac{\partial V(x')}{\partial x'} \times \frac{\partial f(x,u,z')}{\partial x}$$
(3.3)

4 Examples

4.1 Intertemporal job search

Consider the following simplified intertemporal job search model. Suppose that a worker, if unemployed, receives in each period a job offer which promises to pay w forever. If he accepts the offer he receives w in all subsequent periods. Assuming that the worker lives forever, the value of the job offer in period t is

$$\sum_{\tau=t}^{\infty} \beta^{\tau-t} w = \frac{w}{1-\beta}$$

If he rejects the offer, he receives an unemployment compensation c and the chance to receive a new wage offer next period. Wage offers are drawn from a known probability distribution given by $F(w') = \mathbf{P}[w \leq w']$ with F(0) = 0 and F(B) = 1 for some $B < \infty$. Denoting the value of a wage offer by V(w) and assuming that wage offers are independent draws, the value of waiting one more period therefore is

$$c + \beta \int_0^\infty V(w') \mathrm{d}F(w')$$

Thus the value of a wage offer must satisfy the following functional equation:

$$V(w) = \max\left\{\frac{w}{1-\beta}, c+\beta \int_0^\infty V(w') \mathrm{d}F(w')\right\}$$
(4.1)

From figure 3, we see that the solution must have the *reservation wage* property:

$$V(w) = \begin{cases} \frac{\overline{W}}{1-\beta} = c + \beta \int_0^\infty V(w') \mathrm{d}F(w'), & w \le \overline{W}; \\ \frac{w}{1-\beta}, & w \ge \overline{W}. \end{cases}$$
(4.2)

where \overline{W} is called the reservation wage. It is determined through the following equation:

$$\frac{\overline{W}}{1-\beta} = c + \beta \int_0^\infty V(w') dF(w') \Rightarrow \overline{W} - c = \frac{\beta}{1-\beta} \int_{\overline{W}}^\infty (w' - \overline{W}) dF(w')$$
(4.3)
$$\Rightarrow \overline{W} - c = \beta \int_{\overline{W}}^\infty V(w' - \overline{W}) dF(w')$$
(4.4)

where the left hand side represents the cost of searching one more time having a wage offer \overline{W} at hand and where the right hand side is the expected benefit of searching one more time in terms of the expected present value associated with drawing an offer $w' > \overline{W}$.

Manipulating the above equation then leads to an alternative characterization of the reservation wage:

$$\overline{W} - c = \beta(\mathbb{E}w - c) + \beta \int_0^{\overline{W}} F(w') \mathrm{d}w'$$
(4.5)

This characterization immediately shows that an increase in unemployment compensation or a mean-preserving increase in risk causes the reservation wage to rise.²

²The derivation of equations (4.3), (4.4), and (4.5) can be found in Appendix C.



Figure 3: Job search model

The search model can be used to get a simple equilibrium model of unemployment known as the *bathtub model*. In each period, the worker faces a given probability $\alpha \in (0, 1)$ of surviving into the next period. Leaving the remaining parts of the problem unchanged, the worker's Bellman equation becomes:

$$V(w) = \max\left\{\frac{w}{1-\alpha\beta}, \ c+\alpha\beta\int V(w')\mathrm{d}F(w')\right\}.$$
(4.6)

This is essentially the same equation with only the discount factor changing. Let the implied reservation wage be \bar{w} . Assume that in each period there is a constant fraction $1 - \alpha$ of new born workers. They replace an equal number of newly departed workers. If all new workers start out being unemployed, the unemployment rate U_t obeys the law of motion:

$$U_t = (1 - \alpha) + \alpha F(\bar{w}) U_{t-1}.$$
(4.7)

The right hand side is the sum of the fraction of new born workers and the fraction of surviving workers who remained unemployed at the end of last period (i.e. those who rejected offers because they were less than the reservation wage \bar{w}). The steady state unemployment rate U^* is

$$U^* = \frac{1 - \alpha}{1 - \alpha F(\bar{w})}.\tag{4.8}$$

Let N be the length of time until a successful offer is encountered. N = 1means that the first job offer is accepted. If λ denotes the probability that a job offer is rejected, i.e. $\lambda = \int_0^{\bar{w}} dF(w)$, then $\mathbf{P}[N = j] = (1 - \lambda)\lambda^{j-1}$. Thus, the waiting time is geometrically distributed. The mean waiting time then becomes

$$\sum_{j=1}^{\infty} j\mathbf{P}[N=j] = \sum_{j=1}^{\infty} j(1-\lambda)\lambda^{j-1} = \frac{1}{1-\lambda}.$$
(4.9)

Next we show that it is never optimal for the worker to quit. For this purpose consider the following three options:

(A1): accept the wage and keep it forever: $\frac{w}{1-\alpha\beta}$.

(A2): accept the wage and quit after t periods:

$$\frac{w - (\alpha\beta)^t w}{1 - \alpha\beta} + (\alpha\beta)^t \left(c + \alpha\beta \int V(w') dF(w')\right) = \frac{w}{1 - \alpha\beta} - (\alpha\beta)^t \frac{w - \bar{w}}{1 - \alpha\beta}$$

(A3): reject the wage: $c + \alpha\beta \int V(w') dF(w') = \frac{\bar{w}}{1-\alpha\beta}$

If $w < \bar{w}$, then A1 < A2 < A3, and if $w > \bar{w}$, then A1 > A2 > A3. The three alternatives yield the same lifetime utility if $w = \bar{w}$. Thus, A2 is never optimal.

4.2 The Cake Eating Problem

Suppose that you want to eat a cake over the periods t = 0, 1, ..., T. Your intertemporal utility functional V is:

$$V(c_0, c_1, \dots, c_T) = \sum_{t=0}^T \beta^t \ln c_t.$$

The cake has an initial size of $k_0 > 0$. The transition equation for the size of the cake clearly is:

$$k_{t+1} = k_t - c_t$$

The Euler equation for the above optimization problem is given by

$$U'(c_t) = \beta U'(c_{t+1}) \implies c_{t+1} = \beta c_t.$$

The size of the cake in each period can therefore be computed as follows:

$$k_{1} = k_{0} - c_{0}$$

$$k_{2} = k_{1} - c_{1} = \underbrace{k_{0} - c_{0}}_{k_{1}} - \underbrace{\beta c_{0}}_{c_{1}}$$

$$k_{3} = k_{2} - c_{2} = \underbrace{k_{0} - c_{0} - \beta c_{0}}_{k_{2}} - \underbrace{\beta^{2} c_{0}}_{c_{2}} = k_{0} - (1 + \beta + \beta^{2})c_{0} = k_{0} - \frac{1 - \beta^{3}}{1 - \beta}c_{0}$$
...
$$k_{T+1} = k_{0} - \frac{1 - \beta^{T+1}}{1 - \beta}c_{0}$$

From this derivation we see that the Euler equation does not uniquely determine the path of $\{k_t\}$. Only if we add the transversality condition that the cake must be completely eaten in period T, i.e. $k_{T+1} = 0$, can we solve the problem uniquely:

$$c_0 = \frac{1 - \beta}{1 - \beta^{T+1}} k_0.$$

For $T \to \infty$, we get:

$$c_0 = (1 - \beta)k_0$$

which implies

$$c_t = (1 - \beta)k_t.$$

The last equation can be interpreted as the optimal policy rule.

The value of the cake when $T \to \infty$ is thus given by

$$V(k_0) = V(c_0, c_1, \dots) = \sum_{t=0}^{\infty} \beta^t \ln c_t = \sum_{t=0}^{\infty} \beta^t \ln (\beta^t c_0)$$

= $\ln \beta \sum_{t=0}^{\infty} t \beta^t + \ln c_0 \sum_{t=0}^{\infty} \beta^t = \frac{\beta \ln \beta}{(1-\beta)^2} + \frac{\ln c_0}{1-\beta}$
= $\frac{\beta \ln \beta}{(1-\beta)^2} + \frac{\ln((1-\beta)k_0)}{1-\beta}$
= $\frac{\beta \ln \beta}{(1-\beta)^2} + \frac{\ln(1-\beta)}{1-\beta} + \frac{1}{1-\beta} \ln k_0$

4.3 The Neoclassical Growth Model

The simple neoclassical growth model (Ramsey model) has become the work horse of modern macroeconomics. We will start with a simplified deterministic version with inelastic labor supply. For this purpose consider a closed economy whose *production possibilities* are described by a neoclassical production function F in capital k and labor n:

$$F: \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$$
$$(k, n) \longrightarrow y = F(k, n)$$

It is assumed that F is continuously differentiable, strictly monotonically increasing in both arguments, homogenous of degree one (constant returns to scale), and strictly quasi-concave. Moreover, we assume that both capital and labor are essential, i.e. F(0,n) = F(k,0) = 0, and that the Inada conditions hold for capital, i.e. $\lim_{k\to 0} \frac{\partial F(k,n)}{\partial k} = \infty$ and $\lim_{k\to\infty} \frac{\partial F(k,n)}{\partial k} = 0$. The labor force is constant over time and normalized to one: $0 \le n_t \le 1$. The output y_t can either be consumed or invested:

$$c_t + i_t \le y_t = F(k_t, n_t).$$

If output is invested, the capital stock in the next period changes according to the transition equation:

$$k_{t+1} = (1-\delta)k_t + i_t$$

where the depreciation rate $\delta \in [0, 1]$. Thus we have the following resource constraint:

$$c_t + k_{t+1} - (1 - \delta)k_t \le F(k_t, n_t)$$

The economy is populated by a continuum of identical households. The representative household lives forever and derives utility from consumption only according to the additive utility functional:

$$U(c_0, c_1, \dots) = \sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0 < \beta < 1$$

where β is the subjective discount factor. The period utility function U is assumed to be twice continuously differentiable, strictly increasing, and strictly concave with $\lim_{c\to 0} U'(c) = \infty$. Thus the decision problem of the representative household is given by the following intertemporal maximization problem:

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \longrightarrow \max_{\{c_t, k_{t+1}, n_t\}}$$

s.t. $c_t + k_{t+1} - (1-\delta)k_t \leq F(k_t, n_t)$
 $c_t \geq 0, 0 \leq n_t \leq 1, k_t \geq 0$
 $k_0 > 0$ given.

This problem can be simplified through the following arguments:

- There is always a positive value for output because U' > 0. This implies that the resource constraint must hold with equality. Thus choosing k_{t+1} automatically implies a value for c_t given k_t . We can therefore choose k_{t+1} instead of c_t as the control variable.
- Because labor is supplied inelastically and because the marginal product of labor is always positive, $n_t = 1$ for all t.
- Allowing old capital to be retransformed into consumption goods, the resource constraint can be rewritten as

$$c_t + k_{t+1} = F(k_t, 1) + (1 - \delta)k_t = f(k_t)$$

With these modification the above maximization problem can be rewritten as

$$\sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1}) \longrightarrow \max_{\{k_{t+1}\}}$$

s.t. $0 \le k_{t+1} \le f(k_t)$
 $k_0 > 0$ given.

Note that f inherits the properties of F. Given the properties of f, it is easy to see that there exists a maximal sustainable capital stock $k_{\max} > 0$. Setting $c_t = 0$ forever, this maximal capital is determined by equation k = f(k)which has a unique positive solution, given the properties of f. The state variable for this economy is obviously k_t so that the state space is the left open interval $\mathcal{X} = (0, \max\{k_0, k_{\max}\}]$. This space including 0 may be taken to be the control space, i.e. $\mathcal{C} = [0, \max\{k_0, k_{\max}\}]$. The correspondence describing the control constraint is $\Gamma(k_t) = [0, f(k_t)] \subseteq \mathcal{C}$. Alternatively, we can write the maximization problem as a functional equation by applying the dynamic programming technique:

$$V(k) = \sup_{k' \in \Gamma(k)} \left\{ U(f(k) - k') + \beta V(k') \right\}$$

This functional equation can be solved by at least three methods. We illustrate these methods for the case of a logarithmic utility function, a Cobb-Douglas production function, $F(k,n) = Ak^{\alpha}n^{1-\alpha}$, and complete depreciation within the period (i. e. $\delta = 1$). This is one of two specifications for which an explicit analytic solution is available.³

Solution method 1: Value function iteration As will be justified later on, one can approach the solution by iteration. Having found a value function V_j in the *j*-th iteration, the Bellman equation delivers a new value function V_{j+1} :

$$V_{j+1}(k) = \sup_{k' \in \Gamma(k)} \left\{ U(f(k) - k') + \beta V_j(k') \right\}.$$

 $^{^{3}}$ The other specification involves a quadratic objective function coupled with linear constraints (see Section 4.4).

Most importantly, we can start the iteration with an arbitrary function V_0 .

Take, for example, the function $V_0(k) = 0$. Then the Bellman equation simplifies to

$$V_1(k) = \sup_{k' \in \Gamma(k)} \{ U(f(k) - k') \}.$$

The maximization problem delivers $k' = \mu_0(k) = 0$. Inserting the solution into the Bellman equation leads to $V_1(k) = \ln A + \alpha \ln k = v_0^{(1)} + v_1^{(1)} \ln k$.

The equation for V_2 is then given by

$$V_2(k) = \sup_{k' \in \Gamma(k)} \left\{ U(f(k) - k') + \beta V_1(k') \right\}.$$

The first order condition of the maximization problem delivers:

$$\frac{1}{Ak^{\alpha} - k'} = \beta \alpha \frac{1}{k'} \implies k' = \mu_1(k) = \frac{\alpha \beta}{1 + \alpha \beta} A k^{\alpha}.$$

Inserting this optimal value into the objective function yields:

$$V_{2}(k) = \ln \frac{(1 + \alpha\beta)A k^{\alpha} - \alpha\beta A k^{\alpha}}{1 + \alpha\beta} + \beta \ln A + \alpha\beta \ln k'$$
$$= \ln \frac{A}{1 + \alpha\beta} + \beta \ln A + \alpha\beta \ln \frac{A\alpha\beta}{1 + \alpha\beta} + \underbrace{\alpha(1 + \alpha\beta)}_{=v_{1}^{(2)}} \ln k$$
$$= v_{0}^{(2)} + v_{1}^{(2)} \ln k$$

The equation for V_3 is then given by

$$V_3(k) = \sup_{k' \in \Gamma(k)} \left\{ U(f(k) - k') + \beta V_2(k') \right\}.$$

The first order condition for the maximization problem delivers:

$$\frac{1}{Ak^{\alpha} - k'} = \alpha\beta(1 + \alpha\beta)\frac{1}{k'} \implies k' = \mu_2(k) = \frac{\alpha\beta + \alpha^2\beta^2}{1 + \alpha\beta + \alpha^2\beta^2}Ak^{\alpha}.$$

This leads to the following value function V_3 :

$$V_{3}(k) = \beta \ln \frac{A}{1 + \alpha\beta} + \beta^{2} \ln A + \alpha\beta^{2} \ln \frac{A\alpha\beta}{1 + \alpha\beta} + \ln \frac{A}{1 + \alpha\beta + \alpha^{2}\beta^{2}} + \alpha\beta(1 + \alpha\beta) \ln \frac{A\alpha\beta(1 + \alpha\beta)}{1 + \alpha\beta + \alpha^{2}\beta^{2}} + \underbrace{\alpha(1 + \alpha\beta + \alpha^{2}\beta^{2})}_{=v_{1}^{(3)}} \ln k = v_{0}^{(3)} + v_{1}^{(3)} \ln k.$$

The structure of the solutions V_0 , V_1 , V_2 , V_3 , and so on, leads to the conjecture that, by taking j to infinity, the value function is (log-)linear with $V(k) = v_0 + v_1 \ln k$ where v_1 is given by:

$$v_1 = \lim_{j \to \infty} v_1^{(j)} = \frac{\alpha}{1 - \alpha\beta}$$

Similarly, the policy function $k' = \mu_j(k)$ converges to

$$k' = \lim_{j \to \infty} \mu_j(k) = \lim_{j \to \infty} \frac{\sum_{i=1}^j (\alpha \beta)^i}{\sum_{i=0}^j (\alpha \beta)^i} A k^\alpha = \alpha \beta A k^\alpha.$$

Solution method 2: Guess and verifying With this solution method we guess the type of solution and verify that it is indeed a solution. This is a feasible strategy because from theoretical considerations we know that there is a unique solution. Take the guess $V(k) = E + F \ln k$ with constants E and F yet to be determined. The maximization problem then becomes:

$$\ln(A\,k^{\alpha} - k') + \beta V(k') = \ln(A\,k^{\alpha} - k') + \beta E + \beta F \ln k' \longrightarrow \max_{k'}.$$

The first order condition is

$$\frac{1}{Ak^{\alpha} - k'} = \beta F \frac{1}{k'} \quad \Longrightarrow \quad k' = \mu(k) = \frac{F\beta}{1 + F\beta} A \, k^{\alpha}.$$

Inserting this optimal value into the objective function yields:

$$E + F \ln k = \ln \left(Ak^{\alpha} - \frac{F\beta}{1 + F\beta} A k^{\alpha} \right) + E\beta + F\beta \ln \left(\frac{F\beta}{1 + F\beta} A k^{\alpha} \right)$$

This implies

$$E + F \ln k = \ln \frac{A}{1 + F\beta} + \alpha \ln k + E\beta + F\beta \ln \frac{F\beta A}{1 + F\beta} + F\alpha\beta \ln k$$

Equating the coefficients of $\ln k$ leads to following equation for F:

$$F = \alpha + F\alpha\beta \implies F = \frac{\alpha}{1 - \alpha\beta}$$

Thus, $k' = \mu(k) = \alpha \beta A k^{\alpha}$ as before.

Inserting this results into the objective function leads to:

$$(1-\beta)E = \ln(1-\alpha\beta) + \ln A + \frac{\alpha\beta}{1-\alpha\beta}\ln(A\alpha\beta) + \left[\alpha\left(1+\frac{\alpha\beta}{1-\alpha\beta}\right) - \frac{\alpha}{1-\alpha\beta}\right]\ln k$$
$$= \ln(1-\alpha\beta) + \ln A + \frac{\alpha\beta}{1-\alpha\beta}\ln(A\alpha\beta)$$
$$\Longrightarrow$$
$$E = \frac{1}{1-\beta}\left(\ln(1-\alpha\beta) + \ln A + \frac{\alpha\beta}{1-\alpha\beta}\ln(A\alpha\beta)\right)$$

Note that first order condition delivers a first order linear difference equation in k_t :

$$k_{t+1} = \frac{F\beta}{1+F\beta} A k_t^{\alpha} = \alpha\beta A k_t^{\alpha}$$
$$\ln k_{t+1} = \ln A\alpha\beta + \alpha \ln k_t$$

As $0 < \alpha < 1$ this is a stable difference equation.

Solution method 3: Difference equation The first order condition for the maximization problem is:

$$U'(f(k) - k') = \beta V'(k').$$

From the Benveniste-Scheinkman formula we get:

$$V'(k) = U'(f(k) - k')f'(k) - U'(f(k) - k')\frac{\partial k'}{\partial k} + \beta V'(k')\frac{\partial k'}{\partial k}$$
$$= U'(f(k) - k')f'(k).$$

We therefore get the following second order difference equation:

$$U'(f(k) - k') = \beta U'(f(k') - k'')f'(k')$$
$$\frac{1}{Ak_t^{\alpha} - k_{t+1}} = \beta \frac{1}{Ak_{t+1}^{\alpha} - k_{t+2}} A\alpha k_{t+1}^{\alpha - 1}$$
$$\frac{k_{t+1}}{Ak_t^{\alpha} - k_{t+1}} = \frac{A\alpha\beta k_{t+1}^{\alpha}}{Ak_{t+1}^{\alpha} - k_{t+2}}$$
$$\frac{1}{(Ak_t^{\alpha}/k_{t+1}) - 1} = \frac{\alpha\beta}{1 - (k_{t+2}/Ak_{t+1}^{\alpha})}.$$



Figure 4: Ramsey model

If we set $y_{t+1} = \frac{k_{t+2}}{Ak_{t+1}^{\alpha}}$, we get the following non-linear difference equation:

$$y_{t+1} = (1 + \alpha\beta) - \frac{\alpha\beta}{y_t}.$$

This equation admits two steady states: $\alpha\beta$ and 1. The second steady state cannot be optimal because it implies that there is no consumption. As the second steady state is unstable, we must have $y_t = \alpha\beta$ which implies $k_{t+1} = A\alpha\beta k_t^{\alpha}$. The situation is depicted in figure 4

4.4 The Linear Regulator Problem

Another prototypical case is when the objective function is quadratic and the law of motion linear. Consider the following general setup know as the **Optimal Linear Regulator Problem:**

$$V(x_0) = -\sum_{t=0}^{\infty} \beta^t \left(x'_t R x_t + u'_t Q u_t \right) \to \max_{u_t}, \qquad 0 < \beta < 1$$
(4.10)
$$x_{t+1} = A x_t + B u_t, \qquad x_0 \text{ given},$$

where x_t denotes the *n*-dimensional state vector and u_t the *k*-vector of controls. *R* is positive semidefinite symmetric $n \times n$ matrix and *Q* is a positive definite symmetric $k \times k$ matrix. *A* and *B* are $n \times n$, respectively $n \times k$ matrices. Note that the problem has been simplified by allowing no interaction between x_t and u_t .

The Bellman equation can thus be written as

$$V(x_t) = \max_{u_t} -\{(x_t'Rx_t + u_t'Qu_t) + \beta V(x_{t+1})\} \quad \text{s.t.} \quad x_{t+1} = Ax_t + Bu_t,$$

where $V(x_t)$ denotes the value of x in period t. We may solve this equation by guessing that V(x) = -x'Px for some positive semidefinite symmetric matrix P. Using this guess and the law of motion the Bellman equation becomes:

$$-x'Px = \max_{u} -\{x'Rx + u'Qu + \beta(Ax + Bu)'P(Ax + Bu)\}$$

The first order condition of the maximization problem of the right hand side is^4

$$(Q + \beta B'PB)u = -\beta B'PAx$$

Thus, we get the feedback rule:

$$u = -Fx = -\beta(Q + \beta B'PB)^{-1}B'PAx.$$

Inserting this rule into the Bellman equation and rearranging terms leads to

$$P = R + \beta A' P A - \beta^2 A' P B (Q + \beta B' P B)^{-1} B' P A.$$

This equation is known as the *Riccati equation*. It can be solved by iteration:

$$P_{j+1} = R + \beta A' P_j A - \beta^2 A' P_j B (Q + \beta B' P_j B)^{-1} B' P_j A$$

⁴Here we used the following rules for matrix derivatives: $\frac{\partial Ax}{\partial x} = A'$, $\frac{\partial Ax}{\partial x'} = A$, $\frac{\partial x'Ax}{\partial x} = (A + A')x$.

starting from $P_0 = 0$. A sufficient condition for the iteration to converge is that the eigenvalues of A are absolutely strictly smaller than one.

If the optimal rule is inserted into the law of motion, we obtain the *closed-loop solution*:

$$x_{t+1} = (A - BF)x_t$$

This difference equation is stable if the eigenvalues of A - BF are strictly smaller than one in absolute value. There is a large literature investigating the conditions on R, Q, A, and B such a stable closed loop solution obtains. Basically, two conditions must be met. First, A and B must be such that the controller can drive down x_t to zero starting from any initial condition x_0 . Second, the R must be such that the controller wants to drive x_t to zero.

As an illustration consider the following simple numerical example with $\beta = 1$, A = 1, B = 1, Q = 1, and R = 1. It is instructive to note that this specification allows for the possibility that some plans yield a limit to the infinite sum in (4.10) equal to $-\infty$. Such plans are, however, never optimal as they are dominated by plans with a finite limit. The general principles of dynamic programming outlined in Chapter 5 give a theoretical account of this possibility (see in particular Assumption (5.2)). Given the scalar specification, the Riccati equation can be solved analytically:⁵

$$P = 1 + P - \frac{P^2}{1+P} \implies P = (1+\sqrt{5})/2 \approx 1.618$$

The negative solution can be disregarded because we are looking for positive solutions. This implies that

$$u = -Fx = -\frac{P}{1+P}x$$

Inserting this in the law of motion for x_t gives the closed-loop solution:

$$x_{t+1} = \left(1 - \frac{P}{1+P}\right)x_t = \frac{1}{1+P}x_t$$

This solution is stable despite A = 1 and $\beta = 1$. Finally, $V(x) = P x^2 < \infty$.

⁵It can also be shown that the Riccati difference equation is stable although A = 1 and $\beta = 1$.

5 Principles of Dynamic Programming: the deterministic case

As we have seen, many dynamic economic problems can be cast in either of the two following forms: a sequence problem (SP) or a functional (Bellman) equation (FE).

(SP)
$$\sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

s.t. $x_{t+1} \in \Gamma(x_t), x_0$ given

(FE)
$$V(x) = \sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta V(y) \right\}$$

where $0 < \beta < 1$. The optimization problem is thus completely specified $(\mathcal{X}, \Gamma, F, \beta)$ where \mathcal{X} is the state space, Γ a correspondence from \mathcal{X} into \mathcal{X} describing the feasibility constraints, a period return function F defined on A, the graph of Γ (i.e. $\mathbf{A} = \{(x, y) \in \mathcal{X} \times \mathcal{X} : y \in \Gamma(x)\}$), and a subjective discount factor β with $0 < \beta < 1$.

A sequence $\pi = \{x_t\}$ is called a *plan*. A plan is called *feasible* if $x_{t+1} \in \Gamma(x_t)$ for all t. The set of all feasible plan starting in x_0 is denoted by $\Pi(x_0)$. Thus any $\pi \in \Pi(x_0)$ fulfills the constraints of (SP). In order for the problem to make sense, the following two assumptions must hold:

Assumption 5.1. $\forall x \in \mathcal{X}, \Gamma(x) \neq \emptyset$, in particular $\Pi(x_0) \neq \emptyset$. Assumption 5.2. $\forall x_0 \in \mathcal{X} \text{ and } \forall \pi \in \Pi(x_0), \lim_{T \to \infty} \sum_{t=0}^{T} F(x_t, x_{t+1})$ exists. Limit values of ∞ and $-\infty$ are possible.

Note that assumption 5.2 is fulfilled if F is bounded.

For any $T = 0, 1, 2, \ldots$ define the function $U_T : \Pi(x_0) \to \mathbb{R}$ by

$$U_T(\pi) = \sum_{t=0}^T \beta^t F(x_t, x_{t+1})$$

This is just the partial sum of discounted returns for a feasible plan π . Assumption 5.2 allows to define a function $U: \Pi(x_0) \to \mathbb{R}$ by

$$U(\pi) = \lim_{T \to \infty} U_T(\pi)$$

where \mathbb{R} may now include ∞ and $-\infty$. By assumption 5.1, $\Pi(x_0)$ is not empty and the objective function in (SP) is well defined. This allows us to define the supremum function $V^* : \mathcal{X} \to \mathbb{R}$ by

$$V^*(x_0) = \sup_{\pi \in \Pi(x_0)} U(\pi)$$

This function is well defined and is the **unique** function which satisfies:

- (i) If $|V^*(x_0)| < \infty$, $V^*(x_0) \ge U(\pi)$ for all $\pi \in \Pi(x_0)$.
- (ii) If $|V^*(x_0)| < \infty$, $\forall \epsilon > 0 \exists \pi \in \Pi(x) : V^*(x_0) \le U(\pi) + \epsilon$.
- (iii) If $|V^*(x_0)| = \infty$, there exists a sequence of plans $\{\pi_k\}$ such that $U(\pi_k)$ converges to ∞ .
- (iv) If $|V^*(x_0)| = -\infty$, there exists a sequence of plans $\{\pi_k\}$ such that $U(\pi_k)$ converges to $-\infty$.

Thus V^* satisfies the (FE),

(i) if
$$|V^*(x_0)| < \infty$$
, $V^*(x_0) \ge F(x_0, y) + \beta V^*(y)$ for all $y \in \Gamma(x_0)$;

(ii)
$$|V^*(x_0)| = \infty$$
, $\exists \pi_k \in \Gamma(x_0) : \lim_{k \to \infty} F(x_0, y_k) + \beta V^*(y_k) = \infty$;

(iii) $|V^*(x_0)| = -\infty$, $F(x_0, y) + \beta V^*(y) = -\infty$ for all $y \in \Gamma(x_0)$.

Lemma 5.1. If the problem $(\mathcal{X}, \Gamma, F, \beta)$ satisfies assumptions 5.1 and 5.2, then

$$U(\pi) = F(x_0, x_1) + \beta U(\pi') \qquad \forall \pi \in \Pi(x_0) \text{ and } \forall x_0 \in \mathcal{X}$$

where $\pi' = (x_1, x_2, ...).$

Proposition 5.1. If the problem $(\mathcal{X}, \Gamma, F, \beta)$ satisfies assumptions 5.1 and 5.2, V^* satisfies (FE).

Proposition 5.2. If the problem $(\mathcal{X}, \Gamma, F, \beta)$ satisfies assumptions 5.1 and 5.2, then for any solution V to (FE) which satisfies the transversality condition

$$\lim_{t \to \infty} \beta^t V(x_t) = 0$$

for all $\pi \in \Gamma(x_0)$ and for all $x_0 \in \mathcal{X}$, $V = V^*$.

Remark 5.1. Proposition 5.2 implies that the solution to (FE) is unique.

A feasible plan π^* is optimal for x_0 if $U(\pi^*) = V^*(x_0)$.

Proposition 5.3. If the problem $(\mathcal{X}, \Gamma, F, \beta)$ satisfies assumptions 5.1 and 5.2, any optimal plan π^* which attains the supremum in (SP) satisfies

$$V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*).$$

Proposition 5.4. If the problem $(\mathcal{X}, \Gamma, F, \beta)$ satisfies assumptions 5.1 and 5.2, any optimal plan π^* which satisfies

$$V^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_{t+1}^*)$$

and

$$\limsup_{t \to \infty} \beta^t V^*(x_t) \le 0$$

attains the supremum in (SP).

Every nonempty correspondence $\mu : \mathcal{X} \to \mathcal{X}$ with $\mu(x) \subseteq \Gamma(x)$ is called a *policy correspondence*. If $\mu(x)$ is single valued, it is called a *policy function*. The optimal policy correspondence μ^* is defined by

$$\mu^*(x) = \{ y \in \Pi(x) : V^*(x) = F(x, y) + \beta V^*(y) \}.$$

The dynamic programming approach is best understood if one views \mathbb{T}

$$\mathbb{T}V(x) = \sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta V(y) \right\}$$

as an *operator* on some function space. We will use the space of bounded continuous functions from \mathcal{X} into \mathbb{R} , denoted by $\mathfrak{B}(\mathcal{X})$, as the appropriate function space. This space is a Banach space with the supremum norm (i.e. $||f|| = \sup_{x \in \mathcal{X}} f(x)$). Convergence in this space is equivalent to uniform convergence. A solution to (FE) is therefore nothing but a *fixed-point* of this operator in $\mathfrak{B}(\mathcal{X})$:

$$V = \mathbb{T}V$$

Note that all function in $\mathfrak{B}(\mathcal{X})$ satisfy the transversality condition. The idea of using this operator view is the following. Suppose V_0 has some property. Then we ask if $\mathbb{T}V_0$ also has this property. If this is true $\mathbb{T}^t V_0$ has also this property. But then $V = \lim_{t\to\infty} \mathbb{T}^t V_0$ will also share this property because of uniform convergence. In order to make this idea workable, we have to place additional assumption on our decision problem.

Assumption 5.3. $\mathcal{X} \subseteq \mathbb{R}^m$ is convex. Γ is nonempty, compact, and continuous.

Assumption 5.4. $F : \mathbf{A} \to \mathbb{R}$ is bounded and continuous and $0 < \beta < 1$.

Theorem 5.1 (Solution of Bellman Equation). Under assumptions 5.3 and 5.4 the operator \mathbb{T} defined as

$$(\mathbb{T}f)(x) = \sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta V(y) \right\}$$

has the following properties:

- (i) $\mathbb{T}: \mathfrak{B}(\mathcal{X}) \to \mathfrak{B}(\mathcal{X}).$
- (ii) \mathbb{T} has exactly one fixed-point V.
- (*iii*) $\forall V_0 \in \mathfrak{B}(\mathcal{X}), \|\mathbb{T}^t V_0 V\| \leq \beta^t \|V_0 V\|.$
- (iv) The policy correspondence is compact and u.h.c

Assumption 5.5. For all y, F(x, y) is strictly increasing in x.

Assumption 5.6. Γ is increasing, i.e. $x \leq x' \Rightarrow \Gamma(x) \subseteq \Gamma(x')$.

Theorem 5.2 (Properties of the Solution). Under assumptions 5.3, 5.4, 5.5 and 5.6, the solution V to (FE) is strictly increasing.

Assumption 5.7. F(x, y) is concave (with strict inequality if $x \neq x'$).

Assumption 5.8. Γ is convex in the sense that for any θ , $0 \le \theta \le 1$, and $x, x' \in \mathcal{X}, y \in \Gamma(x)$ and $y' \in \Gamma(x')$ implies $\theta y + (1 - \theta)y' \in \Gamma(\theta x + (1 - \theta)x')$.

This assumption excludes increasing returns to scale.

Theorem 5.3 (Further Properties of the Solution). Under assumptions 5.3, 5.4, 5.7, and 5.8 if V satisfies (FE) and the policy correspondence G is well defined then

- (i) V is strictly concave and
- (ii) G is continuous and single-valued.

Theorem 5.4 (Solution found by Iteration). Let $(\mathcal{X}, \Gamma, F, \beta)$ satisfy assumptions 5.3, 5.4, 5.7, and 5.8 and let V satisfy (FE) with a well-defined policy correspondence G. Denote by $\mathfrak{B}'(\mathcal{X})$ the space of bounded, continuous, concave functionals and let $V_0 \in \mathfrak{B}'(\mathcal{X})$. Define $\{(V_n, g_n)\}$ by

- (i) $V_{n+1} = \mathbb{T}V_n$, $n = 0, 1, 2, \dots$, and
- (*ii*) $g_n(x) = \arg \max_{y \in \Gamma(x)} [F(x, y) + \beta V_n(y)].$

Then $\{g_n\}$ converges point wise to g. If \mathcal{X} is compact, the convergence is uniform.

Theorem 5.5 (Differentiability of Solution). Let $(\mathcal{X}, \Gamma, F, \beta)$ satisfy assumptions 5.3, 5.4, 5.7, and 5.8 and let V satisfy (FE) with a well-defined policy correspondence G. In addition let F be continuously differentiable on the interior of the graph of Γ . If $x_0 \in \operatorname{int} \mathcal{X}$ and $g(x_0) \in \operatorname{int} \Gamma(x_0)$, then V is continuously differentiable at x_0 with derivative given by

$$V_i(x_0) = F_i(x_0, g(x_0)), \qquad i = 1, \dots, m.$$

The subscript i denotes the derivative with respect to the i-th element of $x_0 \in \mathbb{R}^m$.

Remark 5.2. If V is twice continuously differentiable, the monotonicity of g could be established by differentiating $F_x(x, g(x)) = bV'(x)$. Although V is continuously differentiable under fairly general conditions, conditions ensuring that V is twice continuously differentiable are quite strong.

6 Principles of Dynamic Programming: the stochastic case

In the following we will consider stochastic dynamic programming problems of the following type:

$$V(x,z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int_Z V(y,z')Q(z,dz') \right\}$$

The major difference is that the agent(s) have to take stochastic shocks into account. This implies that the supremum has to be taken with respect to F plus the expected value of discounted future V. The shocks are governed by a *transition function Q*. Under suitable assumptions on the shocks, the required mathematical properties of the value function V are preserved under integration. Thus, the results for the deterministic model carry over virtually without change.

The state space is now the product space of the measurable spaces $(\mathcal{X}, \mathfrak{X})$ and (Z, \mathfrak{Z}) , i.e. $(\mathcal{S}, \mathfrak{S}) = (\mathcal{X} \times Z, \mathfrak{X} \times \mathfrak{Z})$, describing the possible values of the endogenous and exogenous state variables. Q is a transition function on (Z, \mathfrak{Z}) . The correspondence Γ describing the feasibility constraints is now a correspondence from \mathcal{S} into \mathcal{X} . The graph of Γ is denoted by \mathbf{A} , i.e. $\mathbf{A} = \{(x, y, z) \in \mathcal{X} \times \mathcal{X} \times Z : y \in \Gamma(x, z)\}$. F is again a one period return function on \mathbf{A} and β is the subjective discount factor with $0 < \beta < 1$.

Assumption 6.1. $\mathcal{X} \subseteq \mathbb{R}^m$ is closed and convex with Borel subsets \mathfrak{X} .

Assumption 6.2. One of the following conditions hold:

- (i) Z is a countable set and \mathfrak{Z} is the σ -algebra containing all subsets of Z.
- (ii) Z is a compact and convex subset of \mathbb{R}^m with Borel subsets \mathfrak{Z} and the transition function Q on (Z,\mathfrak{Z}) has the Feller property.

Assumption 6.3. The correspondence $\Gamma : \mathcal{X} \times Z \to \mathcal{X}$ is nonempty, compactvalued and continuous.

Assumption 6.4. The period return function $F : \mathbf{A} \to \mathbb{R}$ is bounded and continuous, and $\beta \in (0, 1)$.

Theorem 6.1 (Solution of the Stochastic Bellman Equation). Let $(\mathcal{X}, \mathfrak{X})$, $(Z, \mathfrak{Z}), Q, \Gamma, F$, and β satisfy assumptions 6.1, 6.2, 6.3, and 6.4. Define the operator \mathbb{T} on $\mathfrak{B}(\mathcal{S})$ by

$$(\mathbb{T}f)(x,z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int_Z f(y,z')Q(z,\mathrm{d}z') \right\}.$$

Then the operator \mathbb{T} has the following properties:

- (i) $\mathbb{T}:\mathfrak{B}(\mathcal{S})\to\mathfrak{B}(\mathcal{S});$
- (ii) \mathbb{T} has a unique fixed point V;

(*iii*)
$$\forall V_0 \in \mathfrak{B}(S), \|\mathbb{T}^n V_0 - V\| \le \beta^n \|V_0 - V\|$$
, $n = 1, 2, ...;$

(iv) The policy correspondence $G: \mathcal{S} \to \mathcal{X}$ defined by

$$G(x,z) = \left\{ y \in \Gamma(x,z) : V(x,z) = F(x,y,z) + \beta \int_Z V(x,z')Q(z,dz') \right\}$$

is nonempty, compact-valued and u.h.c.

Assumption 6.5. $\forall (y,z) \in \mathcal{X} \times Z$, $F(.,y,z) : \mathbf{A}_{y,z} \to \mathbb{R}$ is strictly increasing.

Assumption 6.6. $\forall z \in \mathbb{Z}, \Gamma(., z) : \mathcal{X} \to \mathcal{X} \text{ is strictly increasing, i.e. } x \leq x' \Rightarrow \Gamma(x, z) \subseteq \Gamma(x', z).$

Theorem 6.2 (Property of Solution: Stochastic Case). Let $(\mathcal{X}, \mathfrak{X})$, (Z, \mathfrak{Z}) , $Q, \Gamma, F, and \beta$ satisfy assumptions 6.1, 6.2, 6.3, 6.4, 6.5, and 6.6. If V denotes the fixed point of \mathbb{T} , then, $\forall z \in Z, V(.,z) : \mathcal{X} \to \mathbb{R}$ is strictly increasing.

Assumption 6.7. $\forall z \in Z, F(.,.,z) : \mathbf{A}_z \to \mathbb{R}$ satisfies: $F(\theta(x,y) + (1 - \theta)(x',y'),z) \geq \theta F(x,y,z) + (1-\theta)F(x',y',z), \forall \theta \in (0,1), \forall (x,y), (x',y') \in \mathbf{A}_z$ and the inequality is strict if $x \neq x'$.

Assumption 6.8. $\forall z \in Z \text{ and } \forall x, x' \in \mathcal{X}, y \in \Gamma(x, z) \text{ and } y' \in \Gamma(x', z)$ implies

$$\theta y + (1 - \theta)y' \in \Gamma(\theta x + (1 - \theta)x', z) \text{ for any } \theta \in [0, 1].$$

Theorem 6.3 (Further Properties of Solution: Stochastic Case). Let $(\mathcal{X}, \mathfrak{X})$, $(Z, \mathfrak{Z}), Q, \Gamma, F, and <math>\beta$ satisfy assumptions 6.1, 6.2, 6.3, 6.4, 6.7, and 6.8. If V denotes the fixed point of \mathbb{T} and G the policy correspondence, then, $\forall z \in Z, V(., z) : \mathcal{X} \to \mathbb{R}$ is strictly concave and $G(., z) : \mathcal{X} \to \mathcal{X}$ is a continuous (single-valued) function.

Theorem 6.4 (Solution found by Iteration: Stochastic Case). Let $(\mathcal{X}, \mathfrak{X})$, (Z, \mathfrak{Z}) , Q, Γ, F , and β satisfy assumptions 6.1, 6.2, 6.3, 6.4, 6.7, and 6.8. If V denotes the fixed point of \mathbb{T} and g = G the single-valued policy function, then $V \in \mathfrak{C}'(\mathcal{S})$, the set of bounded continuous functions on \mathcal{S} which are weakly concave jointly in their first m arguments. Moreover, if $V_0 \in \mathfrak{C}'(\mathcal{S})$ and if $\{(V_n, g_n)\}$ is defined by

$$V_n = \mathbb{T}V_{n-1}, n = 1, 2, \dots$$

and

$$g_n(x) = \arg \max_{x \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int_Z V_n(y,z')Q(z,dz') \right\}.$$

Then $\{g_n\}$ converges pointwise to g. If \mathcal{X} and Z are both compact, then the convergence is uniform.

Assumption 6.9. $\forall z \in Z, F(.,.,z)$ is continuously differentiable at $(x, y) \in int \mathbf{A}_z$.

Theorem 6.5 (Differentiability of Solution: Stochastic Case). Let $(\mathcal{X}, \mathfrak{X})$, $(Z, \mathfrak{Z}), Q, \Gamma, F, and <math>\beta$ satisfy assumptions 6.1, 6.2, 6.3, 6.4, 6.7, 6.8, and 6.9. If $V \in \mathfrak{C}'(\mathcal{S})$ denotes the unique fixed point of \mathbb{T} and g = G the single-valued policy function, then $V(., z_0)$ is continuously differentiable in x at x_0 for $x_0 \in \operatorname{int} \mathcal{X}$ and $g(x_0, z_0) \in \operatorname{int} G(x_0, z_0)$ with derivatives given by

$$V_i(x_0, z_0) = F_i(x_0, g(x_0, z_0), z_0), \qquad i = 1, 2, \dots, m.$$

The subscript i denotes the derivative with respect to the i-th element.

7 The Lucas Tree Model

This presentation follows closely the exposition given in Sargent (1987, chapter 3). For details we refer to the original paper by Lucas (1978). Imagine an economy composed of a large number of agents with identical preferences and endowments. Each agent owns exactly one tree which is perfectly durable. In each period, the tree yields a fruit or dividend d_t . The fruit is the only consumption good and is nonstorable. Denote by p_t the price of the tree. Then each agent is assumed to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0 < \beta < 1,$$

subject to

$$A_{t+1} = R_t (A_t - c_t), \qquad A_0 \text{ given.}$$

Denoting the return R_t is given by $R_t = (p_{t+1} + d_{t+1})/p_t$ the Euler equation for this maximization problem is

$$p_t U'(c_t) = \beta \mathbb{E}_t (p_{t+1} + d_{t+1}) U'(c_{t+1}).$$
(7.1)

Because all agents are identical and there is no satiation, every agent just consumes this period's dividends. Thus, we have in equilibrium

$$c_t = d_t.$$

Inserting the equilibrium condition into the Euler equation yields

$$p_t = \beta \mathbb{E}_t \frac{U'(d_{t+1})}{U'(d_t)} (p_{t+1} + d_{t+1})$$
(7.2)

Iterating this equation forward in time and using the law of iterated expectations, we get the solution

$$p_t = \mathbb{E}_t \sum_{j=1}^{\infty} \beta^j \left[\prod_{i=0}^{j-1} \frac{U'(d_{t+i+1})}{U'(d_{t+i})} \right] d_{t+j}$$

which simplifies to

$$p_t = \mathbb{E}_t \sum_{j=1}^{\infty} \beta^j \frac{U'(d_{t+j})}{U'(d_t)} d_{t+j}.$$
(7.3)

The share price is the expected discounted stream of dividends, but with time-varying and stochastic discount factors.

An interesting formula is obtained by taking $U(c) = \ln c$. In this case, the pricing formula (7.3) simplifies to

$$p_t = \mathbb{E}_t \sum_{j=1}^{\infty} \beta^j d_t = \frac{\beta}{1-\beta} d_t.$$

This is a simple asset-pricing function which maps the state of the economy at time t, d_t , into the price of an asset at time t.

In order for the conditional expectation in equation (7.1) to be welldefined, it is necessary to impute to the representative agent a view about the law of motion over time of d_t and p_t . The specification of an actual law of motion for d_t , which agents are supposed to know, and a *perceived pricing* function that maps the history of d_t into p_t implies that a law of motion for p_t has been perceived. Given that \mathbb{E}_t in equation (7.1) is computed using the perceived pricing function, equation (7.1) maps the perceived pricing function into an actual pricing function. The notion of a rational expectation equilibrium is that the actual pricing function equals the perceived pricing function. In the following, we will exploit this notion in more detail.

Suppose that dividends evolve according to a Markov process with timeinvariant transition probability distribution function F(x', x) defined as

$$F(x', x) = \mathbf{P}[d_{t+1} \le x' | d_t = x].$$

Conditional expectations are computed with respect to this law which is supposed to be known by all agents.

Denote by s_t the number of shares held by the agent. The period budget constraint is

$$c_t + p_t s_{t+1} \le (p_t + d_t) s_t.$$

Lifetime utility is again given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0 < \beta < 1.$$

As explained in the previous paragraph, in order for the conditional expectation to be well-defined we must specify a law of motion for the stock price:

$$p_t = h(x_t).$$

This law of motion together with the evolution of x_t defines the perceived law of motion for the tree prices. The Bellman equation can then be written as

$$V(s(h(x) + x)) = \max_{s'} \left\{ U(s(h(x) + x) - h(x)s') + \beta \int V(s'(h(x') + x')) dF(x', x) \right\}.$$
 (7.4)

The first order condition for this optimization problem is

$$h(x)U'(s(h(x) + x) - h(x)s') = \beta \int (h(x') + x')V'(s'(h(x') + x'))dF(x', x).$$

Differentiating the Bellman equation (7.4) with respect to s, we obtain the Benveniste-Scheinkman formula which reads as

$$(h(x) + x)V'(s(h(x) + x)) = (h(x) + x)U'(s(h(x) + x) - h(x)s')$$

or by simplifying

$$V'(s(h(x) + x)) = U'(s(h(x) + x) - h(x)s')$$

Combining the Benveniste-Scheinkman formula and the first-order condition we get

$$h(x)U'(s(h(x) + x) - h(x)s') = \beta \int (h(x') + x')U'(s'(h(x') + x') - h(x')s'')dF(x', x)$$

Defining w(x) by

$$w(x) = h(x)U'(s(h(x) + x) - h(x)s'),$$

we obtain

$$w(x) = \beta \int w(x') dF(x', x) + \beta \int x' U'(s'(h(x') + x') - h(x')s'') dF(x', x)$$

In equilibrium s = s' = s'' = 1 because all agents and all trees are alike so that consumption c(x) = s(h(x) + x) - h(x)s' = x. Thus, we obtain the functional equation in the unknown function w(x):

$$w(x) = \beta \int w(x') \mathrm{d}F(x', x) + \beta \int x' U'(x') \mathrm{d}F(x', x) \tag{7.5}$$

where w(x) = h(x)U'(x). Thus, once w(x) has been determined, the pricing function h(x) can be recovered from w(x) = h(x)U'(x) as U'(x) is known.

Denote by g(x) the function $g(x) = \beta \int x' U'(x') dF(x', x)$, then we can define the operator \mathbb{T} as follows

$$(\mathbb{T}w)(x) = \beta \int w(x') \mathrm{d}F(x', x) + g(x)$$

In order to apply Blackwell's sufficient condition for a contraction (see Theorem B.3), we must show that \mathbb{T} is a continuous mapping from the space of continuous and bounded functionals, $\mathfrak{B}(\mathcal{X})$ with $\mathcal{X} \subseteq \mathbb{R}^n$, to itself. For this purpose, we follow Lucas (1978) and assume that the utility function U(c) is bounded by some number B and concave with U(0) = 0. The definition of concavity of U implies that $U(y) - U(x) \leq U'(x)(y-x)$. Setting y = 0, then leads to $xU'(x) \leq B$. Thus,

$$g(x) = \beta \int x' U'(x') \mathrm{d}F(x', x) \le \beta B < B.$$

Therefore g(x) is also bounded by B. Given some mild technical conditions on the transition function F(x', x), it can be shown that g(x) is also continuous. Thus, \mathbb{T} is continuous. It is then easy to verify that \mathbb{T} is monotone and discounts. Let $w_1, w_2 \in \mathfrak{B}(\mathcal{X})$ such that $w_1(x) \leq w_2(x)$ for all $x \in \mathcal{X}$, then

$$(\mathbb{T}w_1)(x) = \beta \int w_1(x') \mathrm{d}F(x', x) + g(x)$$
$$\leq \beta \int w_2(x') \mathrm{d}F(x', x) + g(x) = (\mathbb{T}w_2)(x).$$

Thus, \mathbb{T} is monotone. Furthermore, for any constant $c \in \mathbb{R}$ we have

$$(\mathbb{T}(w+c))(x) = \beta \int (w+c)(x') dF(x',x) + g(x)$$
$$= \beta \int w(x') dF(x',x) + \beta c \int dF(x',x) + g(x) = (\mathbb{T}w)(x) + \beta c.$$

Thus, \mathbb{T} discounts. Therefore \mathbb{T} is a contraction so that a unique fixed point exists. This fixed point is approached by iterations on w(x), respectively h(x):

$$w^{(j+1)}(x) = \beta \int w^{(j)}(x') \mathrm{d}F(x', x) + g(x),$$

or in terms of h(x)

$$h^{(j+1)}(x)U'(x) = \beta \int h^{(j)}(x')U'(x')dF(x',x) + g(x)$$
(7.6)

Thus, the operator \mathbb{T} maps a perceived pricing function $h^{(j)}$ into an actual pricing function $h^{(j+1)}$. A rational expectation equilibrium is then nothing but a fixed point of this mapping so that the actual pricing function equals the perceived one.

As an example, take $U(x) = \ln x$. With this utility function $g(x) = \beta \int x'(1/x') dF(x', x) = \beta$. Guess that the solution is of the form h(x) = ax. This implies that

$$ax\frac{1}{x} = \beta \int ax'\frac{1}{x'} dF(x', x) + \beta = a\beta + \beta$$

which results in

$$a = \frac{\beta}{1-\beta}$$
 giving the solution $h(x) = \frac{\beta}{1-\beta} x$.

The Lucas tree model highlights a general principle for the construction of asset-pricing models:

(i) Describe the preferences, technology, and endowments of a dynamic economy. Find the equilibrium intertemporal consumption allocation. This is typically the solution of a social planning problem.

- (ii) Open up a specific market for an asset which represents a specific claim on future consumption. Assume no trade restrictions and derive the corresponding Euler equation.
- (iii) Equate the consumption in the Euler equation to the general equilibrium values found in (i). Then derive the associated asset price.

7.1 The Equity Premium Puzzle

A far reaching application of the Lucas tree model has been presented by Mehra and Prescott (1985). They investigate the empirical implications for the risk premium, i.e. the difference between the average return on equities and the risk-free rate. They work with a discrete state-space version of the model. Suppose that there are $n \ge 2$ state of the world which determine the growth rates of dividends:

$$d_{t+1} = x_{t+1}d_t$$

where the gross growth rate x_{t+1} takes one of possible *n* values. i.e. $x_{t+1} \in \{\sigma_1, \ldots, \sigma_n\}$. The transition from one state to the next is governed by a Markov chain given by

$$\mathbf{P}[x_{t+1} = \sigma_j | x_t = \sigma_i] = P_{ij} > 0.$$

Clearly, $\sum_{j=1}^{n} P_{ij} = 1$. Given that all entries P_{ij} are strictly positive the chain is regular and therefore ergodic because for a given number of periods each state of the chain can move to each state of the chain with a positive probability.

The pricing of securities is ex-dividends and ex-interest. Assuming that the utility function is of the constant relative risk aversion type, i.e. $U(c) = \frac{c^{1-\alpha}-1}{1-\alpha}$, $0 < \alpha < \infty$, the application of the Euler equation (7.3) delivers the price of equities:

$$p_t = \mathbb{E}_t \sum_{j=1}^\infty \beta^j \frac{U'(d_{t+j})}{U'(d_t)} \ d_{t+j} = \mathbb{E}_t \sum_{j=1}^\infty \beta^j \frac{d_t^\alpha}{d_{t+j}^\alpha} \ d_{t+j}.$$

The state is given by (d_t, x_t) and $d_{t+j} = d_t x_{t+1} \dots x_{t+j}$. This implies that the price of the equity in state (d, i), p(d, i), is homogeneous in d_t . The Euler

equation (7.2) becomes

$$p(d,i) = \beta \sum_{j=1}^{n} P_{ij}(\sigma_j d)^{-\alpha} [p(\sigma_j d, j) + \sigma_j d] d^{\alpha}.$$

As the price is homogeneous in the dividends, we can write p(d, i) as $p(d, i) = w_i d$ and the above equation can be written as

$$w_i d = \beta \sum_{j=1}^n P_{ij} \sigma_j^{-\alpha} [w_j \sigma_j d + \sigma_j d]$$

which simplifies to

$$w_i = \beta \sum_{j=1}^n P_{ij} \sigma_j^{1-\alpha} [w_j + 1].$$

This is a system of n equations in n unknowns w_i . This equation system can be written in matrix form as

$$w = \beta P \Sigma w + \beta P \Sigma \mathbf{i}$$

which results in the solution

$$w = \beta (I_n - \beta P \Sigma)^{-1} P \Sigma \mathbf{i}$$

where **i** is a vector of ones and where Σ is a diagonal matrix with diagonal entries $\sigma_j^{1-\alpha}$, $j = 1, \ldots, n$.

In order to bring the model to the data, we compute the mean return for equities, R^e , and for the risk-free rate, R^f . Given that the economy is in state *i* the expected return on the equity is

$$R_{i}^{e} = \sum_{j=1}^{n} P_{ij} r_{ij}^{e}, \quad \text{where } r_{ij}^{e} = \frac{p(\sigma_{j}d, i) + \sigma_{j}d - p(d, i)}{p(d, i)} = \frac{\sigma_{j}(w_{j} + 1)}{w_{i}} - 1.$$

Thus, the average return on equity is

$$R^e = \sum_{j=i}^n \pi_i R_i^e$$

where $\pi = (\pi_1, \ldots, \pi_n)$ is the stationary distribution for $P = (P_{ij})$, i.e. π solves $\pi = \pi P$.

The risk-free asset pays one unit of consumption irrespectively of the state. Thus, the price of this asset, $p^{f}(d, i)$ in state (d, i) is

$$p^{f}(d,i) = \beta \sum_{j=1}^{n} P_{ij}(\sigma_j d)^{-\alpha} d^{\alpha} = \beta \sum_{j=1}^{n} P_{ij}(\sigma_j)^{-\alpha}$$

Thus, the return is state i is

$$R_i^f = \frac{1}{p_i^f} - 1.$$

This gives the average return on the risk-free asset as

$$R^f = \sum_{i=1}^n \pi_i R_i^f.$$

In their paper Mehra and Prescott (1985) calibrate this model to the U.S. data over the period 1889 to 1978 setting n = 2. They find

$$P = \begin{pmatrix} 0.43 & 0.57 \\ 0.57 & 0.43 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 1.054 \\ 0.982 \end{pmatrix}.$$

They then compute the combinations of the average risk-free rate and the average risk-premium $R^e - R^f$ for values of the risk aversion parameter α ranging from small positive numbers to a maximum of 10 and for the discount factor β ranging from 0.925 to 0.9999. This delineates an admissible region reproduced in Figure 5. The empirical average over this period for the risk premium is 6.18 percent and 0.80 percent for the risk-free rate. These values are way of the admissible region, the model is clearly incompatible with the data. This result turned out to be very robust and is since then called the *equity premium puzzle*. This puzzle is still not completely resolved and remains an active area of research (see the survey by Mehra and Prescott, 2003).

A Some Topological Concepts

Definition A.1. The set $\mathcal{X} \subseteq \mathbb{R}^n$ is convex iff

$$\theta x + (1 - \theta)y \in \mathcal{X}$$
 for all $x, y \in \mathcal{X}$ and $0 < \theta < 1$.



Figure 5: The Equity Premium Puzzle

Definition A.2. Let f be a function from $\mathcal{X} \subseteq \mathbb{R}^n$ to \mathbb{R} where \mathcal{X} is a convex set. Then f is convex on \mathcal{X} iff

 $f(\theta x + (1 - \theta)y) \le \theta f(x) + (1 - \theta)f(y) \quad \text{for all } x, y \in \mathcal{X} \text{ and } 0 < \theta < 1.$

The function f is said to be strictly convex if the inequality above is strict. The function is said to concave, respectively strictly concave, if the inequality is reversed.

Definition A.3. A compact-valued correspondence $\Gamma : \mathcal{X} \to \mathcal{Y}$ is upper hemi-continuous (u.h.c.) at x if $\Gamma(x)$ is nonempty and if, for every sequence $x_n \to x$ and every sequence $\{y_n\}$ with $y_n \in \Gamma(x_n)$ for all n, there exists a convergent subsequence of $\{y_n\}$ whose limit point y is in $\Gamma(x)$.

Definition A.4. The function $f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ where \mathcal{X} is convex is quasi-concave *iff*

$$f(y) \ge f(x) \Rightarrow f(\theta y + (1 - \theta)x) \ge f(x)$$
 for all $x, y \in \mathcal{X}$ and $0 \le \theta \le 1$.

The function f is said to be strictly quasi-concave iff

$$\begin{split} f(y) \geq f(x) \Rightarrow f(\theta y + (1-\theta)x) > f(x) \\ for \ all \ x, y \in \mathcal{X} \ with \ x \neq y \ and \ 0 \leq \theta \leq 1 \end{split}$$

Remark A.1. f is quasi-concave iff $\{x \in \mathcal{X} : f(x) \ge \alpha\}$ is convex for all $\alpha \in \mathbb{R}$.

Remark A.2. f is concave implies that f is quasi-concave. The converse is not true.

Definition A.5. A metric (distance) on some set \mathcal{X} is a mapping $d : X \times X \to [0, \infty)$ such that

(i) d(x, y) = 0 if and only if x = y (definiteness)

(ii)
$$d(x,y) = d(y,x)$$
 (symmetry)

(iii) $d(x,y) \le d(x,z) + d(z,y)$ (triangle inequality)

for all $x, y, z \in \mathcal{X}$. A metric space is a pair (\mathcal{X}, d) with \mathcal{X} being a set and d being a metric on it.

Definition A.6. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a metric space (\mathcal{X}, d) is called a cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$, *i.e.* if

 $\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ such \ that \ \forall n, m \ge N : d(x_n, x_m) < \epsilon.$

Lemma A.1. Let (\mathcal{X}, d) be a metric space. Then the following assertions hold:

- (i) Each convergent sequence in \mathcal{X} is a Cauchy sequence.
- (ii) Each Cauchy sequence is bounded.
- *(iii)* If a Cauchy sequence has a convergent subsequence, then it converges.

Definition A.7. A metric d on a set \mathcal{X} is called complete if every Cauchy sequence converges. A metric space (\mathcal{X}, d) is called complete if d is a complete metric on \mathcal{X} .

Theorem A.1. The Euclidean metric on \mathbb{R}^n is complete.

B Fixed Point Theorems

Definition B.1. Let (\mathcal{X}, d) be a metric space, then an operator $\mathbb{T} : \mathcal{X} \to \mathcal{X}$ is a contraction mapping (with modulus β) if there exists a $\beta \in (0, 1)$ such that

 $d(\mathbb{T}x,\mathbb{T}y) \leq \beta d(x,y) \quad \text{for all } x, y \in \mathcal{X}.$

Theorem B.1. If \mathbb{T} is a contraction mapping then \mathbb{T} is uniformly continuous on \mathcal{X} , and thus continuous

Proof. For any $\epsilon > 0$ chose $\delta = \epsilon$, then for all x, y with $d(x, y) < \delta$ $d(\mathbb{T}x, \mathbb{T}y) \leq \beta d(x, y) < d(x, y) < \delta = \epsilon$. Thus, \mathbb{T} is uniformly continuous and therefore also continuous.

Theorem B.2 (Contraction Mapping Theorem). If (\mathcal{X}, d) is a complete metric space and $\mathbb{T} : \mathcal{X} \to \mathcal{X}$ is a contraction mapping with modulus β then

- (i) \mathbb{T} has exactly one fixed point $x^* \in \mathcal{X}$;
- (*ii*) For all $x_0 \in \mathcal{X}$, $d(\mathbb{T}^n x_0, x^*) \leq \beta^n d(x_0, x^*)$, $n = 0, 1, 2, \dots$

Proof. UNIQUENESS: Suppose x and y are two fixed points of \mathbb{T} , then

$$0 \le d(x, y) = d(\mathbb{T}x, \mathbb{T}y) \le \beta d(x, y)$$

which implies by $\beta < 1$ that d(x, y) = 0, therefore x = y. EXISTENCE: Fix $x_0 \in \mathcal{X}$ and define x_n as $\mathbb{T}^n x_0$. The the triangular inequality implies for any $k \in \mathbb{N}$

$$d(x_0, x_k) \leq d(x_0, x_1) + d(x_1, x_2) + \dots + d(x_{k-1}, x_k)$$

= $d(x_0, x_1) + d(\mathbb{T}x_0, \mathbb{T}x_1) + \dots + d(\mathbb{T}^{k-1}x_0, \mathbb{T}^{k-1}x_1)$
 $\leq (1 + \beta + \dots \beta^{k-1})d(x_0, x_1) = \frac{1 - \beta^k}{1 - \beta}d(x_0, x_1) \leq \frac{d(x_0, x_1)}{1 - \beta}.$

Hence for any $m, n \in \mathbb{N}$ with m = n + k, we obtain

$$d(x_n, x_m) = d(\mathbb{T}^n x_0, \mathbb{T}^n x_k) \le \beta^n d(x_0, x_k) \le \frac{\beta^n}{1-\beta} d(x_0, x_1).$$



Figure 6: Contraction Mapping

Thus, $\{x_n\}$ is Cauchy sequence because $\beta^n \to 0$ as $n \to \infty$. As \mathcal{X} is a complete metric space, the sequence converges to some $x^* \in \mathcal{X}$, i.e. $\lim_{n\to\infty} x_n = x^*$. Finally, the continuity of \mathbb{T} implies that x^* is fixed point:

$$\mathbb{T}x^* = \mathbb{T}(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} \mathbb{T}x_n = \lim_{n \to \infty} x_{n+1} = x^*.$$

Remark B.1. Let \mathcal{X} be a closed interval [a, b] and d(x, y) = |x - y|. Then $\mathbb{T} : \mathcal{X} \to \mathcal{X}$ is a contraction mapping if for some $\beta \in (0, 1)$,

$$\frac{|\mathbb{T}x - \mathbb{T}y|}{|x - y|} \le \beta < 1, \quad \text{for all } x, y \in \mathcal{X} \text{ with } x \neq y$$

Thus, \mathbb{T} is a contraction mapping if its slope is uniformly smaller than one in absolute value (see Figure 6).

Theorem B.3 (Blackwell's sufficient condition). Let $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathfrak{B}(\mathcal{X})$ the space of bounded functionals with the sup norm. If an operator $\mathbb{T} : \mathfrak{B}(\mathcal{X}) \to \mathfrak{B}(\mathcal{X})$ satisfies

- (i) (monotonicity): $f, g \in \mathfrak{B}(\mathcal{X})$ with $f(x) \leq g(x)$ for all $x \in \mathcal{X}$ implies $\mathbb{T}f(x) \leq \mathbb{T}g(x);$
- (ii) (discounting): There exists $\beta \in (0, 1)$ such that for any function which is constant to some value $c \ge 0$, $[\mathbb{T}(f+c)](x) \le (\mathbb{T}f)(x) + \beta c$, for all $f \in \mathfrak{B}(\mathcal{X}), x \in \mathcal{X}$,

then \mathbb{T} is a contraction with modulus β .

Proof. For any $f, g \in \mathfrak{B}(\mathcal{X}), f \leq g + ||f - g||$. Applying properties (i) and (ii) leads to

$$\mathbb{T}f \le \mathbb{T}(g + \|f - g\|) \le \mathbb{T}g + \beta \|f - g\|.$$

Reversing the role of f and g

$$\mathbb{T}g \le \mathbb{T}(f + \|f - g\|) \le \mathbb{T}f + \beta\|f - g\|.$$

Combining the two inequalities, we obtain $\|\mathbb{T}f - \mathbb{T}g\| \leq \beta \|f - g\|$ as required. \Box

Theorem B.4 (Brouwer's Fixed Point Theorem). Any continuous map of a nonempty compact convex subset of a finite-dimensional normed space into itself has at least one fixed point.

An illustration of Brouwer's fixed point theorem is given in Figure 7 where $f:[a,b] \to [a,b]$. In this simple example it is easy to convince oneself that $\mathbb{T}x$ must cross the 45°-line if we go from the left edge of the square to the right edge. Mathematically speaking, define g(x) = f(x) - x then $g(a) \ge 0$ and $g(b) \le 0$. Hence by the mean value theorem, there exists ξ such that $g(\xi) = 0$ which is equivalent to $f(\xi) = \xi$.

Definition B.2. Let (Z, \mathfrak{Z}) be a measurable space. A transition function is a function $Q: Z \times (Z) \rightarrow [0, 1]$ such that

- (i) Q(z, .) is a probability measure on (Z, \mathfrak{Z}) for all $z \in Z$
- (ii) Q(., A) is a \mathfrak{Z} -measurable function for all $A \in \mathfrak{Z}$.



Figure 7: Illustration of Brouwer's Fixed Point Theorem

Associated to any transition function Q is an operator \mathbb{T} , called *Markov* operator:

$$(\mathbb{T}f)(z) = \int f(z')Q(z, \mathrm{d}z')$$

This operator maps the space of nonnegative, \mathfrak{Z} -measurable, extended realvalued functions into itself. This operator is said to have the *Feller property* if \mathbb{T} is an operator on $\mathfrak{C}(Z)$, the space of bounded continuous functions on Z.

C Derivation of Equations Characterizing the Reservation Wage

In this appendix we derive equations (4.3), (4.4), and (4.5). From equation (4.2) we get

$$\begin{aligned} \frac{\overline{w}}{1-\beta} &= \frac{\overline{w}}{1-\beta} \int_0^{\overline{w}} \mathrm{d}F(w') + \frac{\overline{w}}{1-\beta} \int_{\overline{w}}^{\infty} \mathrm{d}F(w') \\ &= c + \beta \int_0^{\overline{w}} \frac{\overline{w}}{1-\beta} \mathrm{d}F(w') + \beta \int_{\overline{w}}^{\infty} \frac{\overline{w}}{1-\beta} \mathrm{d}F(w') \\ \Longrightarrow \overline{w} \int_0^{\overline{w}} \mathrm{d}F(w') - c &= \frac{1}{1-\beta} \int_{\overline{w}}^{\infty} (\beta w' - \overline{w}) \mathrm{d}F(w') \end{aligned}$$

adding $\overline{w}\int_{\overline{w}}^{\infty}\mathrm{d}F(w')$ on both sides leads to

$$\overline{w} - c = \frac{\beta}{1 - \beta} \int_{\overline{w}}^{\infty} (w' - \overline{w}) \mathrm{d}F(w') = \beta \int_{\overline{w}}^{\infty} V(w' - \overline{w}) \mathrm{d}F(w')$$

These are equations (4.3) and (4.4). Finally,

$$\begin{split} \overline{w} - c &= \frac{\beta}{1 - \beta} \int_{\overline{w}}^{\infty} (w' - \overline{w}) \mathrm{d}F(w') \\ &+ \frac{\beta}{1 - \beta} \int_{0}^{\overline{w}} (w' - \overline{w}) \mathrm{d}F(w') - \frac{\beta}{1 - \beta} \int_{0}^{\overline{w}} (w' - \overline{w}) \overline{d}F(w') \\ &= \frac{\beta}{1 - \beta} \int_{0}^{\infty} (w' - \overline{w}) \mathrm{d}F(w') - \frac{\beta}{1 - \beta} \int_{0}^{\overline{w}} (w' - \overline{w}) \overline{d}F(w') \\ &= \frac{\beta}{1 - \beta} \mathbb{E}w - \frac{\beta}{1 - \beta} \overline{w} - \frac{\beta}{1 - \beta} \int_{0}^{\overline{w}} (w' - \overline{w}) \overline{d}F(w') \\ - (1 - \beta)c &= \beta \mathbb{E}w - \beta \int_{0}^{\overline{w}} (w' - \overline{w}) \overline{d}F(w') \end{split}$$

integrating by parts

 \overline{w}

$$\overline{w} - c = \beta \mathbb{E}(w - c) + \beta \int_0^{\overline{w}} F(w') \overline{d}w'$$

This is equation (4.5).

D Mean Preserving Spreads

Consider a class of distribution functions $F_r(x)$ indexed by r with support included in [0, B], $0 < B < \infty$, and having the same mean. Note that if a random variable X is distributed according to F_r then its mean $\mathbb{E}X$ is given by $\mathbb{E}X = \int_0^B [1 - F_r(x)] \, dx$.⁶ Therefore, if F_r and F_s have the same mean, then

$$\int_0^B \left[F_r(x) - F_s(x)\right] \,\mathrm{d}x = 0.$$

Two distributions F_r and F_s are said to satisfy the single-crossing property if there exists \bar{x} such that

$$F_s(x) - F_r(x) \le 0 \quad (\ge 0) \text{ when } x \ge \bar{x} \quad (x \le \bar{x}).$$

If the two distributions satisfy the above two properties, we can regard F_s as being obtained from F_r by shifting probability mass towards the tail of the distribution keeping the mean constant. The two properties imply

$$\int_0^y \left[F_s(x) - F_r(x) \right] \, \mathrm{d}x \ge 0 \qquad 0 \le y \le B.$$

Rothschild and Stiglitz (1970) regard the move from F_r to F_s as defining a "mean-preserving increase in spread". Figure 8 illustrates the single–crossing property where $F_s(x)$ is viewed as being more riskier than $F_r(x)$.

If F_r is differentiable with respect to r, we can give an alternative characterization of a mean-preserving increase in risk:

$$\int_0^y \frac{\partial F_r(x)}{\partial r} \, \mathrm{d}x \ge 0, \qquad 0 \le y \le B$$

with $\int_0^B \frac{\partial F_r(x)}{\partial r} \, \mathrm{d}x = 0.$

⁶This can be seen applying the integration–by–parts formula: $\int_{a}^{b} (1-F)(x) dx = (1-F(x))|_{a}^{b} - \int_{a}^{b} x d(1-F) = -\int_{a}^{b} x d(1-F) = \int_{a}^{b} x dF = \mathbb{E}X.$



Figure 8: Two distributions satisfying the single-crossing property

E The Riemann-Stieltjes Integral

The usual Riemann integral is defined as

$$\int_{a}^{b} f(x) dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_{i})(x_{i} - x_{i-1})$$

where f is a real and bounded function defined on some interval [a, b]. The x_0, x_1, \ldots, x_n partition the interval [a, b] such that $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. The ξ_i are arbitrary numbers from the interval $[x_{i-1}, x_i]$. Note that the limit does not exist for every function. Thus, there are functions which are not Riemann integrable. Monoton or continuous functions are Riemann integrable.

This definition can be generalized to the so-called Riemann-Stieltjes integral by replacing $(x_i - x_{i-1})$ by $(g(x_i) - g(x_{i-1}))$ where g is some real bounded function on [a, b]:

$$\int_{a}^{b} f(x) dg(x) = \lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_{i}) \left(g(x_{i}) - g(x_{i-1}) \right).$$

The Riemann-Stieltjes integral exists if f is continuous and g is a function of bounded variation.⁷ For g(x) = x, we get the Riemann integral above. Moreover, if g is continuously differentiable with derivative g',

$$\int_{a}^{b} f(x) \mathrm{d}g(x) = \int_{a}^{b} f(x)g'(x) \mathrm{d}x.$$

The Riemann-Stieltjes integral is particularly relevant when g is a cumulative distribution function. Such functions are right-continuous, nondecreasing with existing left-hand limits. Moreover, they can have countably many discontinuity points (jumps) as will be the case, for example, with a discrete probability distribution. Thus, the expectation of a random variable X with distribution function F can be written as

$$\mathbb{E}X = \int_{-\infty}^{\infty} x \mathrm{d}F(x) = \begin{cases} \int_{-\infty}^{\infty} x f(x) \mathrm{d}x, & \text{if } F \text{ has density } f;\\ \sum_{i} x_{i} p_{i}, & p_{i} = \mathbf{P}[X = x_{i}]. \end{cases}$$

In the latter case F is a step function.

⁷A function g on [a, b] is called a function of bounded variation if there exists a constant M > 0 such that for all partitions x_0, x_1, \ldots, x_n of the interval [a, b] $\sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \leq M$. Continuous and monoton functions are of bounded variation.

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