Time –Varying Rational Expectations Models

Klaus Neusser *

December 18, 2018

Abstract

This paper develops a comprehensive theory for rational expectations models with time–varying (random) coefficients. Based on the Multiplicative Ergodic Theorem it develops a "linear algebra" in terms of Lyapunov exponents, defined as the asymptotic growth rates of trajectories. Together with their associated Lyapunov spaces they provide a perfect substitute for the eigenvalue/eigenspace analysis used in constant coefficient models. In particular, they allow the construction of explicit solution formulas similar to the standard case. These methods and their numerical implementation is illustrated using a canonical New Keynesian model with a time–varying policy rule and lagged endogenous variables.

JEL CLASSIFICATION: C02, C61
KEYWORDS: time–varying rational expectations models, Lyapunov exponents, multiplicative ergodic theorem

*I thank Roger Farmer, Sylvia Kaufmann, Dirk Niepelt and Martin Wagner as well as seminar participants at the University of Ljubljana and the Vienna University of Technology for perceptive comments.
Department of Economics, University of Bern, Schanzeneckstrasse 1, P.O. Box 8573, CH-3001 Berne, Switzerland. Email: klaus.neusser@vwi.unibe.ch

1 Introduction

This paper develops a comprehensive theory for rational expectations models with time–varying (random) coefficients. Based the Multiplicative Ergodic Theorem it develops a "linear algebra" in terms of Lyapunov exponents, defined as the asymptotic growth rates of trajectories. Together with their associated Lyapunov spaces they provide a perfect substitute for the eigenvalue/eigenspace analysis used in constant coefficient models. In particular, they allow the construction of explicit solution formulas similar to the case of constant coefficients.

The introduction of Lyapunov exponents is inevitable because the eigenvalues of the coefficient matrices provide in general no information about the dynamic properties of the underlying difference equation. Elaydi (2005, p. 191), Colonius and Kliemann (2014, pp. 109–110), Costa, Fragoso, and Marques (2005, section 3.3.2) and Neusser (2017, appendix A) provide examples where the coefficient matrix alternates between two values such that the model becomes unstable although all eigenvalues in both alternatives are absolutely smaller than one.[1] Hence the spectral theorem (see Meyer, 2000, chapter 7.2) which underlies the usual solution formulas in the case of constant coefficients is no longer applicable. Fortunately, the Lyapunov exponents are a perfect substitute. Taking a *random dynamical systems* perspective, Oseledets' celebrated Multiplicative Ergodic Theorem (MET) lifts the eigenvalue\eigenvector analysis used in the constant coefficient case to the case of stochastically varying coefficients using the Lyapunov exponents\spaces.[2] Thus, the MET paves the way for the derivation of explicit solution formulas for rational expectations model with stochastically varying coefficients. These solution formulas turn out to be in the spirit of Blanchard and Kahn (1980), Klein (2000), and Sims (2001) and are therefore directly interpretable in economic terms.

The theory outlined in this paper turns out to be very versatile and easily implementable. It allows to analyze a wide class of rational expectations models which, so far, have been inaccessible or hardly accessible to economists. It encompasses, in particular, models with Markov–switching or autoregressively moving coefficients with and without lagged dependent variables. Thereby, the exogenous forcing variables follow general stochastic

[1] Francq and Zakoïan (2001) provide further illuminating examples in the context of multivariate Markov-switching ARMA models.

[2] Colonius and Kliemann (2014) provides a clear and accessible presentation of the MET by relating it to the standard eigenvalue\eigenspace analysis. The monograph by Arnold (2003) and Viana (2014) also offer an elaborated and excellent expositions, but are mathematically more demanding.

processes, including ARMA processes. There is, however, a price to pay for this generalization: the Lyapunov exponents and their associated Lyapunov spaces cannot, in general, be computed analytically, but are only accessible via numerical procedures. This alleged disadvantage is compensated by powerful numerical algorithms which do not only allow the computation of the Lyapunov exponents, but also their corresponding Lyapunov spaces (see Dieci and Elia (2008) and Froyland et al. (2013) for details). In doing so, this is the first paper to make explicit use in economics of Lyapunov exponents/spaces along with the MET to analyze dynamic, respectively rational expectations models.

The development of such a theory seems pressing as the presupposition of constant coefficients in rational expectations macroeconomic models is a very tenuous position. Indeed, there are several convincing reasons to believe in time-varying coefficients instead. First, time-varying coefficient models arise naturally from the linearization of nonlinear models along solution paths (Elaydi, 2005, p. 219–220). Second, the relationships describing the economy undergo structural changes giving rise to drifting coefficients as emphasized by Lucas' critique. Sargent (1999), for example, provides an interpretation in terms of self-confirming equilibria and learning. Third, policies and policy rules are subject to change. Cogley and Sargent (2005), Primiceri (2005), or Chen, Leeper, and Leith (2015), among many others, provide empirical evidence with regard to U.S. monetary policy.

Related Literature This paper shares the ambition expressed in the pioneering work by Farmer, Waggoner, and Zha (2009), Farmer, Waggoner, and Zha (2011), and Foerster et al. (2016) to provide a solid and adequate methodology to analyze rational expectations models with time–varying coefficients. Their analysis is, however, limited Markov-switching rational expectations models. This venture triggered a number of papers which extended the original framework to incorporate notably lagged endogenous and predetermined variables (Barthélemy and Marx, 2017; Cho, 2016; Foerster, 2016). They derive conditions for model determinacy and stability based on a mean-square stability criterion as originally proposed by Costa, Fragoso, and Marques (2005) within the context of Markov–switching coefficient models. As deserving this research may be, the framework seems unnecessarily restrictive. First, the focus on second moments excludes many variables of interest to economists such as asset prices. Second, the reliance on finite state Markov-switching mechanisms as a source of time-variation excludes many interesting alternatives, like slowly drifting coefficients. Finally, in contrast to the Lyapunov spectrum (the set of all

Lyapunov exponents) the mean-square stability criterion and its adaptations do not completely capture the dynamic properties of the model. This becomes, however, necessary if one wants to derive and analyze explicit model solutions. This will require to divide the Lyapunov exponents and their associated Lyapunov spaces into those associated with stable, respectively unstable dynamics. The approach presented here will cope with these issues without substantial costs.

There is a related, but apparently disconnected literature on Markov–switching multivariate ARMA models in the spirit of HAMILTON (1989, 2016). This time series literature shows that a strictly stationary solution exists whenever the top–Lyapunov exponent (the largest Lyapunov exponent) is strictly negative (see BRANDT (1986) and BOUGEROL and PICARD (1992)). In addition, FRANCQ and ZAKOÏAN (2001) derive a necessary and sufficient condition for the existence of a second–order stationary solution (i.e. for mean–square stability). Their eigenvalue criterion is practically similar to those provided in CHO (2016) and FOERSTER (2016) (see FRANCQ and ZAKOÏAN, 2001, theorem 2). While the top–Lyapunov exponent is an important characteristic, it does not in general capture all dynamic properties of the underlying model. In this manner, the time series literature bridges the gap between the dynamical systems approach advocated in this paper and the macro economic oriented literature from the previous paragraph.

The paper proceeds by first exposing the general setup in Section 2. It lays out the basic assumptions, presents and explains the Multiplicative Ergodic Theorem (MET). The main contribution of the paper is presented in Section 3 which derives explicit solution formulas together with some implications. Having presented the general theory, Section 4 illustrates the usefulness of the proposed methods by applying them to a simple New Keynesian model with a randomly switching Taylor rule. More specifically, I allow the policy to switch between two states which would lead to determinate, respectively indeterminate models in the deterministic case. The economic relevance and the consequences of these two alternative rules have been analyzed by GALÍ (2011). This model also serves as a prime example in the contributions cited above. Finally, I draw some conclusions for further applications and research.

2 Random Coefficients Rational Expectations Models

2.1 Model Setup

The class of rational expectations models with time-varying (random) coefficients analyzed in this paper consist of an affine state equation which describes the evolution of the state $x_t \in \mathbb{R}^d$ over time subject to boundary conditions. More specifically, the state equation takes the following form:

$$\mathbb{E}_t x_{t+1} = \psi_t(x_t) = A_t x_t + b_t, \qquad t \in \mathbb{Z}, \tag{2.1}$$

where ψ_t is a randomly chosen affine map.[3] In contrast to conventional rational expectations models, the coefficient matrix A_t is not constant, but varies randomly over time. For convenience and to avoid unnecessary technical intricacy, $A_t \in \mathbb{GL}(d)$, the set of nonsingular $(d \times d)$ matrices, for all t. The term $b_t \in \mathbb{R}^d$ captures all exogenous forces or shocks which impinge on the economy. If the coefficient matrix A_t is constant, the expectational difference equation (2.1) encompasses most of the rational expectations models encountered in the literature. Their properties have been exhaustively analyzed in Blanchard and Kahn (1980), Klein (2000), and Sims (2001) to name just the most relevant references.

The above setup is quite general and versatile. By enlarging the state space, it also encompasses models with lagged endogenous variables which have proven technically difficult to analyze up to now (Cho, 2016). Moreover, the process governing the evolution of A_t can be quite involved. For example, regime-switching and autoregressive schemes are easily implemented. The only practically important feature is that the evolution of A_t is exogenous to the model, i.e. determined outside the model, and that it is taken into account in forming conditional expectations. In the example treated in Section 4 the evolution of A_t is governed by a hidden Markov chain which makes the model a Markov-switching rational expectations model. A similar remark can be made with respect to b_t. In particular, b_t can follow some autoregressive process.

The sequence $\{(A_t, b_t)\}$ of random variables is defined on some probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, i.e. a set Ω endowed with a σ–algebra $\mathfrak{F}$ and a probability measure $\mathbf{P}$. Define $\mathfrak{F}_t$ as $\mathfrak{F}_t = \sigma\{(x_s, A_s, b_s) : s \leq t\}$, i.e. $\mathfrak{F}_t$ is the smallest

[3] Often rational expectations models are written as $B_{1,t} x_t = B_{2,t} \mathbb{E}_t x_{t+1} + z_t$. However, this is equivalent to the form (2.1) with $A_t = B_{2,t}^{-1} B_{1,t}$ and $b_t = -B_{2,t}^{-1} z_t$, provided $B_{2,t} \in \mathbb{GL}(d)$. An extension to singular $B_{2,t}$ matrices is possible, but has been deliberately left out in this paper.

σ–algebra such that (x_s, A_s, b_s) is measurable for all $s \leq t$. The sequence of σ–algebras $\mathfrak{F}_t$ then becomes a filtration adapted to $\{x_t\}$ and $\{(A_t, b_t)\}$ with $\mathfrak{F}_t \subseteq \mathfrak{F}$. $\mathbb{E}_t x_{t+1}$ denotes the conditional expectation with respect to $\mathfrak{F}_t$, i.e. $\mathbb{E}_t x_{t+1} = \mathbb{E}[x_{t+1} \mid \mathfrak{F}_t]$.

From a conceptual point of view it is important to have a clear understanding on the randomness driving $\{A_t\}$. More precisely, I think of $\{A_t\}$ as being driven by a *dynamical system* in the following way. Define the measurable map

$$\theta : \Omega \to \Omega$$

to be the *time shift* and denote its t-fold application by θ^t. Clearly, θ satisfies the cocycle properties: $\theta^0 = \mathrm{Id}$ and $\theta^{t+s} = \theta^t \theta^s$, for all $t, s \in \mathbb{Z}$. Hence, θ is invertible. The time shift is also measure preserving, i.e. $\theta\mathbf{P} = \mathbf{P}$. Maps with these properties are called *metric dynamical systems* in the corresponding literature.[4] In addition, the time shift is ergodic (see f.e. Coudène, 2016, chapter 2). The following assumption summarizes these properties.

Assumption 1 (Time Shift)**.** *The dynamic properties of $\{A_t\}$ are governed by a metric dynamical system $\theta : \Omega \to \Omega$. We specify θ to be the time shift. Hence θ is an invertible, ergodic, and measure–preserving transformation satisfying the cocycle properties.*

In this vein, $A_t(\omega)$ is denoted by $A(\theta^t\omega)$, similarly for ψ_t and b_t. Sometimes, we suppress the dependence on ω and just write A_t, respectively b_t and ψ_t for short. In Section 4.1 we give an explicit account of θ and specify $\{A_t\}$ as the outcome of a hidden Markov chain.

Following Arnold (2003, chapter 1), the evolution of the system on the bundle $\Omega \times \mathbb{R}^n$ can be envisioned as in Figure 1. While ω is shifted by θ to $\theta\omega$, the point x_0 in the fiber $\omega \times \mathbb{R}^d$ is shifted to $x_1 = \psi(\omega)x_0 = A(\omega)x_0 + b(\omega)$ in the fiber $\theta\omega \times \mathbb{R}^d$. In the next period $\theta\omega$ is shifted to $\theta^2\omega$ whereas x_1 is shifted to $x_2 = \psi(\theta\omega) = A(\theta\omega) + b(\theta\omega)$ and so on. Thus, on each fiber the system is affine in the usual sense.

Throughout the paper I assume that the following integrability condition holds.

Assumption 2 (Integrability)**.**

$$\log^+ \|A\|, \quad \log^+ \|A^{-1}\| \quad \textit{and} \ \log^+ \|b\| \in \mathrm{L}^1(\Omega, \mathfrak{F}, \mathbf{P}).$$

Thereby $\log^+ x$ stands for $\max\{\log x, 0\}$. This integrability assumption will be satisfied if the random variables would be essentially bounded. Hence,

[4] For interpretations of θ other than the time shift see Arnold (2003) who presents further details and generalizations.

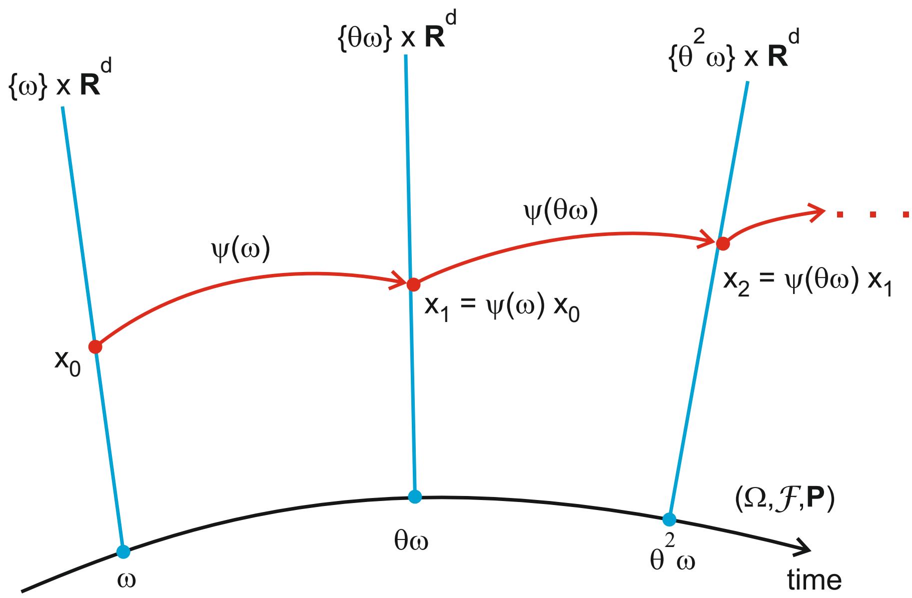

Figure 1: The Evolution of an Affine Random Dynamic System

the integrability condition is satisfied if $\mathbb{E}\|A\|$ and $\mathbb{E}\|b\|$ exist. This is, from a practical point of view, a very weak condition which allow $\{A_t\}$ and $\{b_t\}$ to follow a wide variety of stochastic processes, including autoregressive and Markov–switch ones.

Here and in the following $\|.\|$ denotes the operator norm induced by the Euclidean metric, i.e. $\|A\| = \max_{\|x\|=1} \|Ax\| = \delta_1$ where δ_1 is the largest singular value of A, i.e. δ_1 is the positive square root of the largest eigenvalue of $A'A$. Because all norms are equivalent in $\mathbb{R}^d$, the integrability assumption and the Multiplicative Ergodic Theorem (MET) presented below are independent from any specific submultiplicative norm.

Boundary conditions The model setup is completed by assuming some boundary conditions. They usually come in two forms: initial value and boundedness conditions. The former can be written as

$$c = Rx_0, \qquad c \in \mathbb{R}^r, \tag{2.2}$$

where R is a given $(r \times d)$ matrix of rank r, $0 \leq r \leq d$. In its simplest and most widely used form $R = (I_r, 0)$ which fixes the first r elements of x_0 to be equal to c. When $r < n$, the initial value conditions are not sufficient to pin down x_0 uniquely. Hence, they are complemented by boundedness conditions:

$$\text{there exists } M \in \mathbb{R} \text{ such that } \|x_t\| < M \text{ for all } t \in \mathbb{Z}. \tag{2.3}$$

The latter conditions are usually rationalized by assuming b_t to be bounded which, depending on the particular model in mind, implies that unbounded x_t is economically not feasible or sensible.

2.2 Preliminary Considerations

The expectational difference equation (2.1) satisfies the *superposition principle*: given two solutions $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$, then $\{x_t^{(1)} - x_t^{(2)}\}$ satisfies the linear expectational difference

$$\mathbb{E}_t x_{t+1} = A_t x_t. \tag{2.4}$$

Thus, every solution is of the form:

$$x_t = x_t^{(g)} + x_t^{(p)}$$

where $\{x_t^{(g)}\}$ denotes the general solution of the linear equation (2.4) and $\{x_t^{(p)}\}$ a particular solution of equation (2.1). Hence, the solution can be found in two steps. First, find the general solution to the linear equation (2.4) and then look for a particular solution of equation (2.1).[5]

In order to find the general solution to the linear expectational difference equation (2.4) define $\Phi(t) = \Phi(t, \omega)$ as the random matrix product:

$$\Phi(t) = \Phi(t, \omega) = \begin{cases} A_{t-1}(\omega) \dots A_1(\omega) A_0(\omega), & t = 1, 2, \dots; \\ I_d, & t = 0; \\ A_t(\omega)^{-1} \dots A_{-1}(\omega)^{-1}, & t = -1, -2, \dots \end{cases}$$

This defines a *linear cocycle* over θ, i.e. $A : \Omega \to \mathbb{GL}(n)$ is measurable, $\Phi(0, \omega) = I_d$, and $\Phi(t + s, \omega) = \Phi(t, \theta^s \omega)\Phi(s, \omega)$ for all $t, s \in \mathbb{Z}$ and all $\omega \in \Omega$. Hence, $I_d = \Phi(0, \omega) = \Phi(t, \theta^{-t}\omega)\Phi(-t, \omega)$ which implies the following Lemma by direct calculation.

Lemma 1. *The cocycle properties imply*

$$\begin{aligned} \Phi(-t, \omega) &= \Phi(t, \theta^{-t}\omega)^{-1} \\ \Phi(t, \omega)^{-1} &= \Phi(-t, \theta^t \omega). \end{aligned}$$

Next define a new variable y_t as $y_t = \Phi(t)^{-1} x_t$. It is easy to see that $\{y_t\}$ is a martingale:

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t \left(\Phi(t+1)^{-1} x_{t+1} \right) = \Phi(t+1)^{-1} \mathbb{E}_t x_{t+1} = \Phi(t+1)^{-1} A_t x_t = y_t.$$

Similarly, the time reversed process $\tilde{y}_t = y_{-t}$, $t \in \mathbb{Z}$, is also a martingale. This implies without any additional assumptions that there exists a random variable y satisfying $\lim_{t \to \infty} y_t = y$ a.s. and in mean (see Grimmett and Stirzaker, 2001, section 12.7). Moreover, the original martingale can be reconstructed from y by setting $y_t = \mathbb{E}(y \mid \mathfrak{F}_t)$. Thus, the space of martingales can be continuously parameterized by the space of random variables which are measurable with respect to $\mathfrak{F}$ where $\mathfrak{F} = \sigma\left(\bigcup_{t \in \mathbb{Z}} \mathfrak{F}_t\right)$.[6] The general solution of the linear equation (2.4) can therefore be represented as

$$x_t = (A_{t-1} \dots A_1 A_0)\, x = \Phi(t) x, \tag{2.5}$$

where x is some random variable measurable with respect $\mathfrak{F}$.

Given some starting value x and a realisation ω, the equation (2.5) determines one particular trajectory or solution. Hence, trajectories are parameterized by x and ω and are denote by by $x_t = \varphi(t, \omega, x)$. The existence and the stability properties of the solutions (2.5) depend crucially on the convergence

[5] Cho (2016) follows a similar, but not exactly equal, two step procedure.

[6] Compare this to Klein (2000, Definition 4.3 and Assumption 4.2)

of the matrix products $\Phi(t)$.[7] To investigate this issue, consider a new trajectory obtained from a small perturbation Δx of x. The resulting change Δx_t at time t in the trajectory is then given by $\Delta x_t = \varphi(t, \omega, \Delta x) = \Phi(t, \omega)\Delta x$. This motivates to define the *Lyapunov exponent* λ as the mean exponential rate of divergence or convergence of the two trajectories for $t \to \infty$ and $\|\Delta x\| \to 0$. In the univariate case, this amounts to consider λ defined as $|\Delta x_t| \approx e^{t\lambda}|\Delta x|$. Hence the Lyapunov exponent may be expressed as

$$\lambda(\omega, x) = \lim_{t\to\infty} \lim_{|\Delta x|\to 0} \frac{1}{t} \log \frac{|\Delta x_t|}{|\Delta x|} = \lim_{t\to\infty} \frac{1}{t} \sum_{j=0}^{t-1} \log |a_j|.$$

The above definition describes the mean asymptotic rate of divergence (convergence) from (to) the zero steady state of a small perturbation of the zero solution. To account for the possibility that the limit may not exist, the Lyapunov exponent is generally defined in terms of the limes superior:

$$\lambda(\omega, x) = \limsup_{t\to\infty} \frac{1}{t} \log \|\varphi(t, \omega, x)\| \qquad (2.6)$$

where in the multivariate case the absolute values is replaced by some norm. The Lyapunov exponent therefore describes the asymptotic exponential growth rate of the linear random dynamical system $x_{t+1} = A_t x_t$ with initial value $x \neq 0$. Readers not familiar with Lyapunov exponents should consult Appendix A which shows the relation to the eigenvalues in the case of constant coefficient.

It will be the subject of Oseledets' theorem to show that there exist only a finite number $\ell \leq n$ of different Lyapunov exponents and that they are actually obtained as (double-sided) limits. Moreover, they are independent of x and ω. The importance of the Lyapunov exponents derives from their ability to replace the eigenvalue/eigenspace analysis which provides no information in the case of time-varying coefficients. Indeed, as noted in the Introduction, the eigenvalues of the "time frozen" or "local" matrices A_t are uninformative about the stability of the underlying system of difference equations.[8]

2.3 Oseledets' Multiplicative Ergodic Theorem

Although the eigenvalue/eigenspace analysis of the "time frozen" matrices A_t provides in general no information about the stability and asymptotic behav-

[7] The study of random matrix products has a long history going back to Bellman (1954) and culminated in the acclaimed theorems by Furstenberg and Kesten (1960) and, more relevant for this paper, Oseledets' Multiplicative Ergodic Theorem (MET) (Oseledets, 1968).

[8] See also footnote 1 for references.

ior of the underlying difference equation[9], the Lyapunov exponents/spaces provide a perfect substitute. Indeed Oseledets' Multiplicative Ergodic Theorem (MET) (Oseledets, 1968) lifts the results from standard linear algebra which underlies the constant coefficient case to dynamical systems with random coefficients. An extensive exposition of this theorem with proofs and technical details can be found in Arnold (2003) and Viana (2014). Here I follow Colonius and Kliemann (2014) and present a more accessible version.[10]

Theorem 1 (Multiplicative Ergodic Theorem (MET))**.** *Let θ be a dynamical system satisfying assumption 1 and assume the integrability condition 2 to hold. Then the linear random coefficient dynamical system $x_{t+1} = A_t x_t$ induces a splitting of $\mathbb{R}^d$ into $\ell \leq d$ linear subspaces $L_j(\omega)$, $j = 1, \dots, \ell$. These subspaces have the following properties:*

(i) There is a decomposition (splitting)

$$\mathbb{R}^d = L_1(\omega) \oplus \cdots \oplus L_\ell(\omega)$$

of $\mathbb{R}^d$ into ℓ random subspaces $L_j(\omega)$. These subspaces are not constant, but depend measurably on ω. However, their dimensions remain constant and equal to d_j. The spaces $L_j(\omega)$ are called Lyapunov spaces.

(ii) The Lyapunov spaces are equivariant, i.e. $A(\omega)L_j(\omega) = L_j(\theta\omega)$.

(iii) There are real numbers $\infty > \lambda_1 > \cdots > \lambda_\ell > -\infty$ such that for each $x \in \mathbb{R}^d \setminus \{0\}$ the Lyapunov exponent $\lambda(\omega, x) \in \{\lambda_1, \dots, \lambda_\ell\}$ exists as a limit and

$$\lambda(\omega, x) = \lim_{t \to \pm\infty} \frac{1}{t} \log \|\varphi(t, \omega, x)\| = \lambda_j \text{ if and only if } x \in L_j(\omega) \setminus \{0\}.$$

(iv) The limit

$$\Upsilon(\omega) = \lim_{t \to \infty} \left(\Phi(t, \omega)'\Phi(t, \omega)\right)^{1/2t} \tag{2.7}$$

exists as a positive definite matrix. The different eigenvalues of $\Upsilon(\omega)$ are constants and can be written as $\exp(\lambda_1) > \cdots > \exp(\lambda_\ell)$; the corresponding random eigenspaces are $L_1(\omega), \dots, L_\ell(\omega)$.

[9] See the references in the Introduction and footnote 1.

[10] It is not the most general formulation as there are versions of this theorem with one-sided time (i.e. $\mathbb{N}$ instead of $\mathbb{Z}$), continuous time, and possibly non-invertible matrices A_t.

(v) *The Lyapunov exponents are obtained as limits from the singular values* σ_k *of* $\Phi(t, \omega)$ *as follows. The set of indices* $\{1, 2, \ldots, d\}$ *can be decomposed into subsets* S_j, $j = 1, \ldots, \ell$, *such that for all* $k \in S_j$,

$$\lambda_j = \lim_{t \to \infty} \frac{1}{t} \log \delta_k(\Phi(t, \omega)).$$

It is worth emphasizing that, although the Lyapunov subspaces $L_j(\omega)$ are random as they depend on ω, their dimension remains constant and equal to d_j. Moreover, they are, in general, not orthogonal to each other. Note also the equivariance (invariance or covariance) of these subspaces, i.e. $A(\omega)L_j(\omega) = L_j(\theta\omega)$. This property becomes relevant in the numerical implementation. There exists an alternative decomposition of $\mathbb{R}^d$ into orthogonal subspaces. These subspaces are, however, no longer equivariant (see Froyland et al., 2013, for details). Appendix B provides further explanations of the MET.

3 Construction of Solutions

Given these preliminaries, it is now possible to construct explicit solution formulas. Divide, for this purpose, the state space into three subspaces $L^s(\omega)$, $L^c(\omega)$, and $L^u(\omega)$ corresponding to the Lyapunov spaces with negative, zero, and positive Lyapunov exponents:

$$L^s(\omega) = \bigoplus_{\lambda_j < 0} L(\lambda_j, \omega), \quad L^c(\omega) = L(0, \omega), \text{ and } L^u(\omega) = \bigoplus_{\lambda_j > 0} L(\lambda_j, \omega).$$

These subspaces are called the stable subspace, the center, and the unstable subspace, respectively. Thus, the zero solution of $x_{t+1} = A_t x_t$ is *asymptotically stable* if and only if all Lyapunov exponents are negative, or equivalently if the top Lyapunov exponent (the largest Lyapunov exponent) is negative.[11] This is equivalent to $L^s(\omega) = \mathbb{R}^d$ for some (hence for all) ω. The difference equation (2.1) is called *hyperbolic* if $L^c(\omega) = \emptyset$ or, equivalently, if all Lyapunov exponents are different from zero. For a hyperbolic difference equation the zero solution is called a *saddle point* if both $L^s(\omega)$ and $L^u(\omega)$ have dimensions $d^s = \dim L^s(\omega)$, respectively $d^u = \dim L^u(\omega)$, strictly greater than zero.

In the following, I assume the difference equation to be hyperbolic.

[11] In the time series analysis only the size of the top Lyapunov exponent determines whether a stationary solution exists (Francq and Zakoïan, 2001). In the context of rational expectations models, the whole Lyapunov spectrum (all Lyapunov exponents) is needed to derive an explicit solution formula.

Assumption 3 (Hyperbolicity)**.** *The linear state equation* (2.4) *is hyperbolic, i.e.* $L^c = \emptyset$.

The hyperbolicity assumption becomes particularly relevant when the difference equation is obtained from the linearization of a nonlinear system. Then the Hartman–Grobman theorem tells us that, under the assumption of hyperbolicity, the stability properties of the nonlinear system can be inferred from those of the linearized one (see f.e Robinson (1999, chapter 5) or Coudène (2016, chapter 8) and Arnold (2003, section 4.2.1 and chapter 7) for the random coefficient case). The hyperbolicity assumption also becomes relevant when constructing a particular solution.

Next define $\pi^s(\omega) : \mathbb{R}^d \to L^s(\omega)$ as the projection onto $L^s(\omega)$ along $L^u(\omega)$ and $\pi^u(\omega) : \mathbb{R}^d \to L^u(\omega)$ as the projection onto $L^u(\omega)$ along $L^s(\omega)$. These projections depend on ω because the Lyapunov spaces are random. These projections can be written in terms of matrices (Meyer, 2000, chapter 2.9):

$$\pi^s(\omega) = B(\omega) \begin{pmatrix} I_{d^s} & 0 \\ 0 & 0 \end{pmatrix} B^{-1}(\omega) \quad \text{and} \quad \pi^u(\omega) = B(\omega) \begin{pmatrix} 0 & 0 \\ 0 & I_{d^u} \end{pmatrix} B^{-1}(\omega)$$

where $B(\omega)$ is a basis of $\mathbb{R}^d$ obtained from the union of the basis of $L^s(\omega)$ and $L^u(\omega)$. Although the Lyapunov spaces vary implying varying basis $B(\omega)$, the dimensions d^s and d^u are fixed as stated by the MET.[12] Moreover, the equivariance of the Lyapunov spaces (see (ii) in Theorem 1) implies that $\pi^s(\theta^t\omega)\Phi(t,\omega) = \Phi(t,\omega)\pi^s(\omega)$ and similarly for $\pi^u(\omega)$.

In analogy to the deterministic case, I'm now in a position to construct a particular solution using the towering property of conditional expectations. This leads to the following theorem which is similar in spirit to the analysis of Blanchard and Kahn (1980), Klein (2000) or Sims (2001).

Theorem 2 (Solution Formula)**.** *The rational expectations model consisting of the difference equation* (2.1) *and the boundary conditions* (2.2) *and* (2.3) *admits a unique solution of the form*

$$x_t(\omega) = \Phi(t,\omega)x(\omega) + x_t^{(b)}(\omega) + x_t^{(f)}(\omega) \tag{3.1}$$

where

$$\begin{aligned} x_t^{(b)}(\omega) &= \quad \Phi(t,\omega) \sum_{j=0}^{\infty} \Phi(t-j,\omega)^{-1} \pi^s(\theta^{t-j}\omega) b_{t-1-j} \\ x_t^{(f)}(\omega) &= -\Phi(t,\omega)\ \mathbb{E}_t \left[\sum_{j=0}^{\infty} \Phi(t+j+1,\omega)^{-1}\ \pi^u(\theta^{t+j+1}\omega) b_{t+j} \right] \end{aligned}$$

[12]Compare this to the case of triangular matrices discussed in Appendix B.

if assumptions 1 2, and 3 hold and if the rank condition

$$\operatorname{rank}\begin{pmatrix} R \\ \begin{pmatrix} 0 & I_{d^u} \end{pmatrix} B(\omega)^{-1} \end{pmatrix} = d \tag{3.2}$$

is satisfied. In this case, $x(\omega)$ *is uniquely determined from the equation system* $c - Rx_0^{(p)} = Rx(\omega)$ *and* $\pi^u(\omega)x(\omega) = 0$.

Proof. The proof is relegated to Appendix C. □

As in the constant coefficient case, the part corresponding to the negative Lyapunov exponents is solved backwards in time whereas the part corresponding to positive Lyapunov exponents is solved forward in time. The two parts, therefore, have the interpretation of present values with time-varying "discount factors". The projections applied to b_{t-j-1} and b_{t+j} guarantee that discounting is applied only in the directions where the infinite sum converges. In practice these expressions can be computed recursively. A special case arises if all Lyapunov exponents are positive.

Corollary 1. *If all Lyapunov exponents are positive, the unique solution is given by*

$$x_t = -\Phi(t,\omega)\, \mathbb{E}_t \left[\sum_{j=0}^{\infty} \Phi(t+j+1,\omega)^{-1}\, b_{t+j} \right]. \tag{3.3}$$

Proof. If all Lyapunov exponents are positive, $d^u = d$ and $\pi^u(\omega) = I_d$. In this situation, the rank condition implies that a unique solution can only arise if there is no initial condition. Hence, $x_0(\omega) = 0$ because $\pi^u(\omega)x_0(\omega) = x_0(\omega) = 0$. □

Corollary 2. $r = d^s$ *is a necessary condition for the existence of a unique nonexplosive solution.*

If $r < d^s$, there exists a whole family of nonexplosive solutions and the system is then called *indeterminate*. If $r > d^s$, the equation system is overdetermined and no nonexplosive solution exists.

Another immediate consequence of Theorem 2 is that $x_t^{(p)} = x_t^{(b)} + x_t^{(f)}$ induces an invariant distribution. Hence, $x_t^{(p)}$ behaves like a "moving steady".

4 The New Keynesian Model with Random Policies

4.1 Specification of the Dynamical System

For this approach to be of practical use, it is important to understand how the Lyapunov exponents depend on the underlying randomness and whether they are robust with regard to perturbations. In particular, one wants to know whether this dependence is continuous. While this is a delicate issue in general which is the subject of current research (see VIANA (2014, chapters 9 and 10) and BACKES, BROWN, and BUTLER (2015)), continuity is guaranteed for the application envisaged in this paper.

More specifically, I view A_t to be the outcome of a Markov chain with finite state space $S = \{1, 2, \ldots, s\}$. To each state $j \in S$ I associate a matrix $A^{(j)} \in \mathbb{GL}(d)$. The probability space Ω is then given by sequences of states over $\mathbb{Z}$, i.e. $\Omega = S^{\mathbb{Z}} = \{\omega \mid \omega = (\omega_j)_{j\in\mathbb{Z}} \text{ with } \omega_j \in S\}$, and the time shift θ is defined as $\theta\omega = \theta(\omega_j) = (\omega_{j+1})$.

The probability measure $\mathbf{P}$ on Ω is the Markov measure associated with the transition matrix P where

$$(P)_{ij} = \mathbf{P}[A(\theta\omega) = A^{(j)} \mid A(\omega) = A^{(i)}], \qquad i, j = 1, \ldots, s.$$

Thereby $(P)_{ij}$ is the probability of moving from i to state j in the next period. I assume the Markov chain defined through P to be regular, i.e. P is irreducible (ergodic) and aperiodic. Hence, there exists a unique stationary distribution δ satisfying $\delta' P = \delta'$, $\delta \gg 0$, and $\lim_{t\to\infty} \delta_0' P^t = \delta'$ for any initial distribution δ_0. For $s = 2$, MALHEIRO and VIANA (2015) showed that the Lyapunov exponents depend continuously on the parameters in $A^{(1)}$ and $A^{(2)}$, and P. Moreover, this is equivalent to the continuity with respect to the corresponding Lyapunov spaces (BACKES and POLETTI, 2017).[13] With this specification, the model becomes a Markov–switching rational expectations model which is the prime example analyzed in the literature so far. Hence, the results presented below are comparable to those presented there. However, it is worth emphasizing that the approach based on the MET is very versatile and can account for many other $\{A_t\}$ processes (f.e. autoregressive processes).

In the following example I consider a Markov-switching specification with just two states where the state transition matrix is taken to be

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \qquad p, q \in (0, 1).$$

[13] I thank Jairo Bochi and Anthony Quas for pointing this out to me and indicating me the relevant literature.

As $p, q \in (0,1)$, the chain is regular with invariant distribution $\delta' = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$. Thus, δ is the unique distribution which satisfies $\delta' P = \delta'$. Hence, the chain is on average in $q/(p+q)$ percent of the time in state one and $p/(p+q)$ percent of the time in state two. The mean exit time from state i is $1/(1-(P)_{ii})$ which equals $1/p$ for state one and $1/q$ for state two. For $p = q = 1/2$ the chain has no memory and the sequence of states is iid. Following SHORROCKS (1978), I define a mobility index $M(P)$ as

$$M(P) = \frac{d - \mathrm{tr}P}{d-1} = p + q.$$

This index is equal to the reciprocal of the harmonic mean of the mean exit times.[14] It can be interpreted as measuring the randomness or mobility of the chain.

As there is in general no analytical solution for the Lyapunov exponents, numerical computations have to be applied. Because of the exponential growth, numerical computations are not a straightforward task. A naive application quickly results in numerical overflows. I therefore make use of the iterative QR procedure outlined in DIECI and ELIA (2008). Computations of the Lyapunov spaces are more involved and numerically sensitive (FROYLAND et al., 2013). The Appendix D provides more details.

4.2 Specification of the Model

I take the canonical New Keynesian model (NK model) as my prime example. This model has been extensively analyzed in the literature.[15] Papers most closely related to this one are LUBIK and SCHORFHEIDE (2004), DAVIG and LEEPER (2007), FARMER, WAGGONER, and ZHA (2009), GALÍ (2011), CHEN, LEEPER, and LEITH (2015), FOERSTER (2016), and CHO (2016). A simple version of this model typically consists of the following three equations:

$$\begin{aligned}
y_t &= \mathbb{E}_t y_{t+1} - \sigma^{-1}(i_t - \mathbb{E}_t \pi_{t+1}) + u_t^d, && \text{(IS-equation)} \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa y_t + u_t^s, && \text{(forward-looking Phillips-curve)} \\
i_t &= \phi_t \pi_t, && \text{(Taylor-rule)}
\end{aligned}$$

where the endogenous variables y_t, π_t, and i_t denote income (output gap), the rate of inflation and the nominal interest rate. u_t^d and u_t^s are exogenous

[14] The index is actually conceived by SHORROCKS (1978) for stochastic matrices with quasi dominant diagonals. This aspect is, however, irrelevant for our purposes.

[15] See GALÍ (2018) for a recent assessment of the model.

demand and supply shocks, respectively. In this simplified version of the NK model there are no initial conditions.

As is well-known, the determinacy of the model depends on the strength with which the monetary authority reacts to inflation. For values of ϕ below one, the model becomes indeterminate, whereas for values above one there exists a unique solution. However, monetary policy rules are not constant over time and can often, at least for some time, be characterized as policies with $0 \leq \phi < 1$ (see Galí, 2011, for details and economic interpretations). Hence, it makes sense to treat the parameter ϕ as random and index it by t. This sensitivity of the qualitative nature of the model with respect to ϕ makes the NK model an interesting object of demonstration.

The model can be expressed in terms of $x_{t+1} = (y_{t+1}, \pi_{t+1})'$ by inserting the Taylor-rule in the IS-equation to obtain an affine random coefficient expectational difference equation of the form (2.1):

$$\mathbb{E}_t x_{t+1} = A_t x_t + b_t, \quad t \in \mathbb{Z}, \tag{4.1}$$

where

$$A_t = \frac{1}{\beta} \begin{pmatrix} \beta + (\kappa/\sigma) & (\beta\phi_t - 1)/\sigma \\ -\kappa & 1 \end{pmatrix} \quad \text{and} \quad b_t = \begin{pmatrix} u_t^d - u_t^s/(\beta\sigma) \\ u_t^s/\beta \end{pmatrix}.$$

I consider the case with two states. In the first state the central does not react to inflation at all so that $\phi_t = 0$.[16] In the second state the central bank reacts to inflation with intensity $\phi > 0$. Hence,

$$A^{(1)} = \frac{1}{\beta} \begin{pmatrix} \beta + (\kappa/\sigma) & -1/\sigma \\ -\kappa & 1 \end{pmatrix} \quad \text{and} \quad A^{(2)} = \frac{1}{\beta} \begin{pmatrix} \beta + (\kappa/\sigma) & (\phi\beta - 1)/\sigma \\ -\kappa & 1 \end{pmatrix}.$$

Because $\det A_t = \beta^{-1}(1 + \phi_t\kappa/\sigma) > 1$, $A_t \in \mathbb{GL}(2)$ irrespective of the value of ϕ_t. Both matrices are the same except for the term $(A_t)_{12}$ which is $-1/(\beta\sigma)$ in state one and $(\phi\beta - 1)/(\beta\sigma)$ in state two. I specify the values for the parameters β, κ, and σ to equal 0.985, 0.8, and 1, respectively. The value for ϕ ranges from 0 to 4 in steps of length 0.01. Finally, I examine two alternative transition matrices whose properties are summarized in Table 1.

The MET implies that there are two, not necessarily different, Lyapunov exponents $\lambda_{\max}$ and $\lambda_{\min}$, $\lambda_{\max} \geq \lambda_{\min}$. Moreover, the last assertion of the MET implies $\lambda_{\max} + \lambda_{\min} = \mathbb{E} \log |\det A(\omega)|$. As the determinant of $A(\omega)$ is always strictly greater than one, irrespective of ω, the sum of both Lyapunov exponents is always strictly greater than zero. Hence, $\lambda_{\max} > 0$. The model

[16] This case also occur if the central bank bases its policy on an inflation forecast which takes the interest path as given (Galí, 2011).

Table 1: Characteristics of Transition Matrices

specification	invariant distribution	mean exit times from state 1	from state 2	mobility index $M(P)$
$p = 0.5, q = 0.5$	$(0.5, 0.5)'$	2	2	1
$p = 0.6, q = 0.4$	$(0.4, 0.6)'$	1.66	2.5	1

is therefore determinate if both Lyapunov exponents are greater than zero. In this case, $L^u(\omega) = \mathbb{R}^n$ and the unique solution is given by equation (3.3). If, however, $\lambda_{\min}$ is smaller than zero, the model becomes indeterminate. L^s and L^u are one dimensional and thus nontrivial in this case. Because the model lacks any initial value conditions, the computation of Lyapunov spaces is unnecessary.

4.3 Simulation Results

The simulation results are summarized in the bifurcation diagram in Figure 2. Consider first the benchmark of a deterministic policy (red line). For values of ϕ below one, $\lambda_{max} > 0 > \lambda_{\min}$ implying an indeterminate model. As the reaction of central bank to inflation increases as reflected by larger values of ϕ, $\lambda_{\min}$ increases and crosses the zero line when $\phi = 1$. For values of ϕ bigger than one, both Lyapunov exponents are positive so that the model becomes determinate with a unique solution given by equation (3.3). For $1 < \phi < 1.22$, the model has two distinct Lyapunov exponents corresponding to two distinct real eigenvalues. For values of ϕ greater than 1.22, the eigenvalues become conjugate complex and hence the Lyapunov exponents collapse.

When the policy is no longer fixed, but random, a qualitatively similar picture emerges. Consider first the case $p = q = 0.5$ (green line) so that the states alternate in an iid fashion with mean exit time from each state being equal to two periods. Hence, the central bank reacts on average only in 50 percent of the time to inflation. As shown in Figure 2, there are two distinct Lyapunov exponents. Because the economy spends some time in a state one with no reaction to inflation, the central bank must react more strongly in the state where it takes inflation into account. According to our simulation result, it must react with an intensity ϕ greater than 2.43 in state two to obtain a determinate model. When the central bank follows the anti-inflation policy more often as reflected by the specification $p = 0.6, q = 0.4$, the reaction to

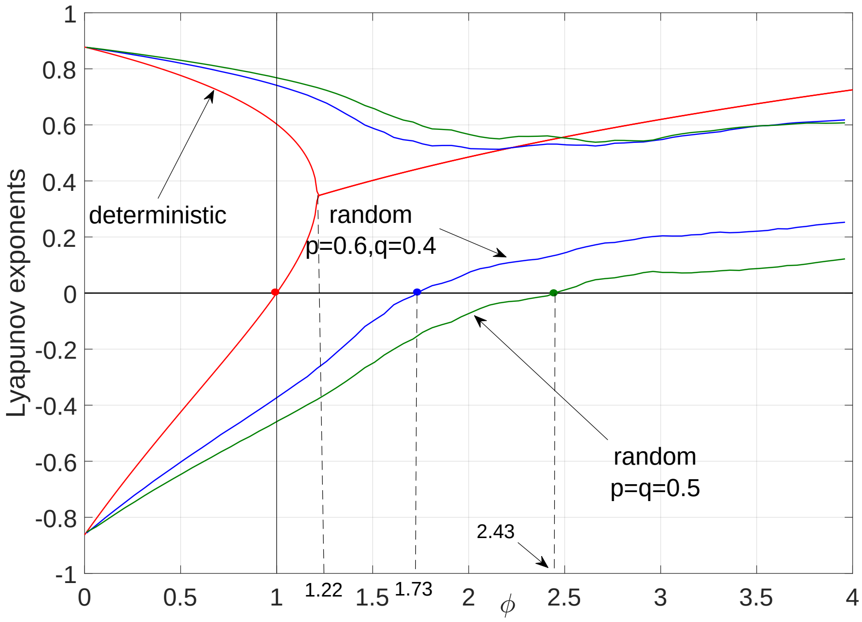

Figure 2: Lyapunov exponents of the New Keynesian model without lagged interest rate as a function of ϕ

inflation in state two can be lower. The model already becomes determinate for values of ϕ higher than 1.73 (blue line). This exercise clearly delineates a trade-off between a strong reaction to inflation in state two and the time spent in state two.

In a next step I generalize the New Keynesian model above by allowing for lagged endogenous variables. Following CHO (2016), I include a lagged interest rate and a monetary disturbance u_t^m in the Taylor rule:

$$i_t = (1-\rho)\phi\pi_t + \rho i_{t-1} + u_t^m, \qquad 0 < \rho < 1.$$

This generalization can be easily accommodated within the theoretical framework presented before by enlarging the state vector by i_t. Hence, $x_{t+1} = (y_{t+1}, \pi_{t+1}, i_t)'$ and the system matrices are given as

$$A_t = \begin{pmatrix} 1 & -(\beta\sigma)^{-1} & \sigma^{-1} \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\kappa & 1 & 0 \\ 0 & (1-\rho)\phi_t & \rho \end{pmatrix}$$

and

$$b_t = \begin{pmatrix} -1 & (\beta\sigma)^{-1} & \sigma^{-1} \\ 0 & -\beta^{-1} & 0 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} u_t^d \\ u_t^s \\ u_t^m \end{pmatrix}$$

Note that $A_t \in \mathbb{GL}(3)$, irrespective of the value of ϕ_t, because $\det A_t = \rho/\beta > 0$. Because of this increase in the dimension of the state vector, the number of Lyapunov exponents rises to 3. As the value of i_t is given and known in period t, the modified model has effectively one initial condition. Hence, Corollary 2 implies that a determinate model requires one negative and two positive Lyapunov exponents.

Leaving the values of all parameters as before and setting $\rho = 0.7$ produces the results summarized in Figure 3. In this figure I have for comparison purposes omitted to display the third Lyapunov exponent. This exponent is always negative and delivers no additional information concerning the determinateness of the model. As one can deduce from Figure 3, the inclusion of a lagged interest rate in the Taylor rule does not alter the qualitative features of the model.

5 Conclusion

The purpose of this paper was to present to economists the mathematical tools which would enable them to analyze rational expectations models with

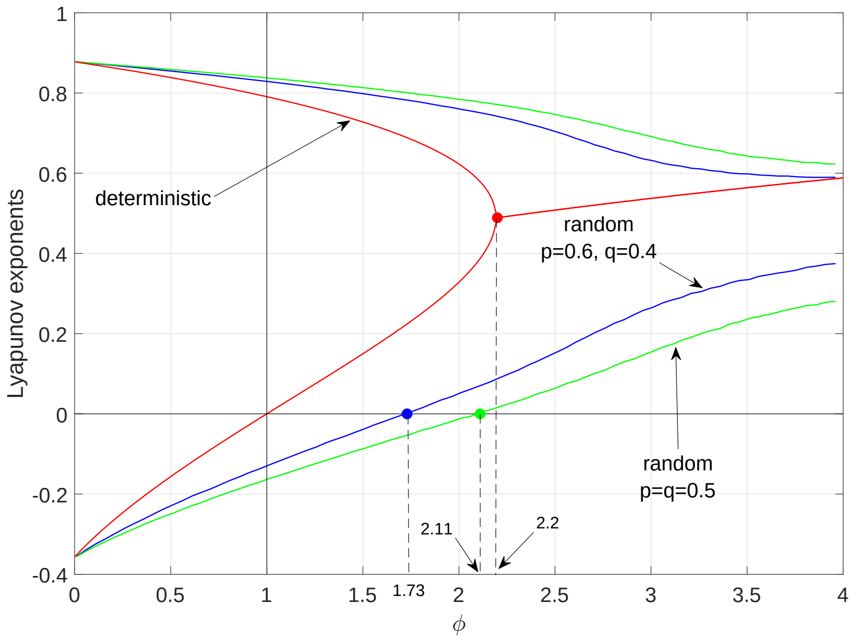

Figure 3: Lyapunov exponents of the New Keynesian model with lagged interest rate as a function of ϕ

time-varying coefficients. The theoretical core of this methodology evolves around the concept of Lyapunov exponents which measure the asymptotic growth rates of trajectories. The Multiplicative Ergodic Theorem by Oseledets then showed that the Lyapunov exponents play a similar role in the analysis of the stability of random dynamical systems as the eigenvalues do in the standard case of constant coefficients. Based in this insight, the paper shows how to construct solutions and analyze the stability of rational expectations models with time–varying coefficients. This approach brings the paper close to the spirit of the standard Blanchard–Kahn analysis of rational expectations models with constant coefficients. The methodology is also relevant for the analysis of regime–switching time series models á la Hamilton (1989, 2016). In this literature, however, the emphasis is on the top Lyapunov exponent whose negativity guarantees the existence of a stationary solution (see Brandt (1986), Bougerol and Picard (1992), and Francq and Zakoïan (2001)).

The application of these tools requires numerical methods as analytical solutions are almost never available. Fortunately, powerful procedures to estimate the Lyapunov exponents as well as the corresponding Lyapunov spaces have been developed (see Dieci and Elia (2008) and Froyland et al. (2013), f.e.). Finally, the paper runs a simulation exercise of a prototype New Keynesian model with random Taylor rule and a lagged endogenous variable to demonstrate the practical usefulness of the approach. This exercise demonstrated that there are no conceptual obstacles to apply this methodology to more sophisticated models.

References

Arnold, Ludwig (2003), *Random Dynamical Systems*, corrected second printing edn., Berlin: Springer-Verlag.

Arnold, Ludwig and Hans Crauel (1992), "Iterated function systems and multiplicative ergodic theorem", in *Stochastic Flows*, M.A. Pinsky and V. Wihstutz, eds., vol. II of *Diffusion Processes and Related Problems in Analysis*, pp. 283–305, Boston: Birkhäuser.

Backes, Lucas, Aaron Brown, and Clark Butler (2015), "Continuity of Lyapunov exponents for cocycles with invariant holonomies", Preprint arXiv:1507.08978v2.

Backes, Lucas and Mauricio Poletti (2017), "Continuity of Lyapunov exponents is equivalent to continuity of Oseledets subspaces", *Stochastics and Dynamics*, 17, 1750047.

Barthélemy, Jean and Magali Marx (2017), "Solving endogenous regime switching models", *Journal of Economic Dynamics and Control*, 77, 1–25.

Bellman, Richard (1954), "Limit theorems of non–commutative operations", *Duke Mathematical Journal*, 21, 491–500.

Berger, Marc A. (1993), *An Introduction to Probability and Stochastic Processes*, New York: Springer–Verlag.

Blanchard, Olivier J. and Charles M. Kahn (1980), "The solution of linear difference models under rational expectations", *Econometrica*, 48, 1305–1311.

Bougerol, Philippe and Nico Picard (1992), "Stationarity of GARCH processes and some nonnegative time series", *Journal of Econometrics*, 52, 115–127.

Brandt, Andreas (1986), "The stochastic equation $Y_{n+1} = A_n Y_n + B_n$ with stationary coefficients", *Advances in Applied Probability*, 18, 211–220.

Chen, Xiaoshan, Eric L. Leeper, and Campbell Leith (2015), "US monetary and fiscal policies – conflict or cooperation?", Working Paper 2015–16, Business School – Economics, University of Glasgow.

Cho, Seonghoon (2016), "Sufficient conditions for determinacy in a class of Markov-switching rational expectations models", *Review of Economic Dynamics*, 21, 182–200.

Cogley, Timothy and Thomas J. Sargent (2005), "Drifts and volatilities: Monetary policies and outcomes in the post WWII US", *Review of Economics Dynamics*, 8, 262–302.

Colonius, Fritz and Wolfgang Kliemann (2014), *Dynamical Systems and Linear Algebra*, vol. 158 of *Graduate Studies in Mathematics*, Providence, Rhode Island: American Mathematical Society.

Costa, Oswaldo Luiz Valle do, Marcelo Dutra Fragoso, and Ricardo Paulino Marques (2005), *Discrete-Time Markov Jump Linear Systems*, Probability and Its Applications, London: Springer.

Coudène, Yves (2016), *Ergodic Theory and Dynamical Systems*, London: Springer Verlag.

Davig, Troy and Eric M. Leeper (2007), "Generalizing the Taylor principle", *American Economic Review*, 97, 607–635.

Dieci, Luca and Cinzia Elia (2008), "SVD algorithms to approximate spectra of dynamical systems", *Mathematics of Computers in Simulation*, 79, 1235–1254.

Elaydi, Saber N. (2005), *An Introduction to Difference Equations*, 3rd edn., New York: Springer.

Farmer, Roger E. A., Daniel F. Waggoner, and Tao Zha (2009), "Understanding Markov-switching rational expectations models", *Journal of Economic Theory*, 144, 1849–1867.

Farmer, Roger E. A., Daniel F. Waggoner, and Tao Zha (2011), "Minimal state variable solutions to Markov-switching rational expectations models", *Journal of Economic Dynamics and Control*, 35, 2150–2166.

Foerster, Andrew, Juan F. Rubio-Ramírez, Daniel F. Waggoner, and Tao Zha (2016), "Perturbation methods for Markov-switching dynamic stochastic general equilibrium models", *Quantitative Economics*, 7, 637–669.

Foerster, Andrew T. (2016), "Monetary policy regime switches and macroeconomic dynamics", *International Econmic Review*, 57, 211–229.

Francq, Christian and Jean-Michel Zakoïan (2001), "Stationarity of multivariate Markov–switching ARMA models", *Journal of Econometrics*, 102, 339–364.

Froyland, Gary, Thorsten Hüls, Gary P. Morriss, and Thomas M. Watson (2013), "Computing covariant Lyapunov vectors, Oseledets vectors, and dichotomy projectors: A comparative numerical study", *Physica D*, 247, 18–39.

Furstenberg, Harry and Harry Kesten (1960), "Products of random matrices", *Annals of Mathematical Statistics*, 31, 457–469.

Galí, Jordi (2011), "Are central banks' projections meaningful?", *Journal of Monetary Economics*, 58, 537–550.

Galí, Jordi (2018), "The state of new Keynesian economics: A partial assessment", *Journal of Economic Perspectives*, 32, 87–112.

Grimmett, Geoffrey and David Stirzaker (2001), *Probability and Random Processes*, 3rd edn., Oxford University Press.

Hamilton, James D. (1989), "A new approach to the economic analysis of nonstationary time series and business cycle", *Econometrica*, 57, 357–384.

Hamilton, James D. (2016), "Macroeconomic regimes and regime shifts", in *Handbook of Macroeconomics*, John B. Taylor and Harald Uhlig, eds., vol. 2, chap. 3, pp. 163–201, Elsevier Press.

Klein, Paul (2000), "Using the generalized Schur form to solve a multivariate linear rational expectations model", *Journal of Economic Dynamics and Control*, 24, 1405–1423.

Lubik, Thomas A. and Frank Schorfheide (2004), "Testing for indeterminacy: An application to U.S. monetary policy", *American Economic Review*, 94, 190–219.

Malheiro, Elas C. and Marcelo Viana (2015), "Lyapunov exponents of linear cocycles over Markov shifts", *Stochastics and Dynamics*, 15(03), 1550020, URL http://dx.doi.org/10.1142/S0219493715500203.

Meyer, Carl D. (2000), *Matrix Analysis and Applied Linear Algebra*, Philadelphia: Society for Industrial and Applied Mathematics.

Neusser, Klaus (2017), "Time-varying rational expectations models: Solutions, stability, numerical implementation", Discussion Paper DP-1701, Departement of Economics, University of Bern.

Oseledets, Valery I. (1968), "A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems", *Transactions of the Moskow Mathematical Society*, 19, 197–231.

Primiceri, Giorgio E. (2005), "Time-varying structural vector autoregressions and monetary policy", *Review of Economic Studies*, 72, 821–852.

Robinson, Clark (1999), *Dynamical Systems*, 2nd edn., Boca Raton, Florida: CRC Press, Taylor & Francis Group.

Sargent, Thomas J. (1999), *Conquest of American Inflation*, Princeton, New Jersey: Princeton University Press.

Shorrocks, Anthony F. (1978), “The measurement of mobility”, *Econometrica*, 46, 1013–1024.

Silva, César Ernesto (2008), *Invitation to Ergodic Theory*, vol. 42 of *Student Mathematical Library*, Providence, Rhode Island: American Mathematical Society.

Sims, Christopher A. (2001), “Solving linear rational expectations models”, *Computational Economics*, 20, 1–20.

Viana, Marcelo (2014), *Lectures on Lyapunov Exponents*, Cambridge Studies in Advanced Mathematics, Cambridge: Cambridge University Press, URL http://dx.doi.org/10.1017/CBO9781139976602.

A Lyapunov Exponents and Eigenvalues

For a better understanding of subsequent arguments it is instructive to examine the case of constant coefficients first. Thus, $x_t = \varphi(t, \omega, x) = \varphi(t, x) = A^t x$ and the Lyapunov exponents are just the logarithms of the distinct moduli of the eigenvalues of A (Colonius and Kliemann, 2014, section 1.5). Denote the different Lyapunov exponents by $\lambda_1 > \cdots > \lambda_\ell$. Then, to each Lyapunov exponent λ_j there is associated a subspace L_j, called *Lyapunov space*, defined as $L_j = \bigoplus E_k$ where the direct sum is taken over all real generalized eigenspaces E_k related to eigenvalues μ_k with $\lambda_j = \log |\mu_k|$. The state space $\mathbb{R}^d$ can then be decomposed into a direct sum of these Lyapunov spaces:

$$\mathbb{R}^d = L_1 \oplus \cdots \oplus L_\ell.$$

The definition of eigenvalues and eigenvectors imply that the Lyapunov spaces are invariant with respect to A, i.e. $AL_j = L_j$ for $j = 1, \ldots, \ell$. This property is called *equivariance.*

Moreover, for any solution $\varphi(t, x)$ with $x \neq 0$

$$\lambda(x) = \lim_{t \to \pm\infty} \frac{1}{t} \log \|\varphi(t, x)\| = \lambda_j \text{ if and only if } x \in L_j.$$

The above characterization of Lyapunov exponents and Lyapunov spaces requires to take the double-sided limit. To see this, suppose, for the sake of the argument, that $d = 2$, $A = \mathrm{diag}(\mu_1, \mu_2)$ with $\mu_1 \neq \mu_2 \in \mathbb{R}$ and $|\mu_1| > |\mu_2|$. Furthermore, let $\|.\|$ be the max–norm and denote $x = (x_1, x_2)'$. The Lyapunov exponents are therefore computed as

$$\lambda(x) = \limsup \frac{1}{t} \log \max\{|\mu_1^t x_1|, |\mu_2^t x_2|\}.$$

In this case, the Lyapunov exponents are $\lambda_1 = \log |\mu_1|$ and $\lambda_2 = \log |\mu_2|$ with the corresponding Lyapunov spaces $L(\lambda_1) = \mathrm{span}(1, 0)'$ and $L(\lambda_2) = \mathrm{span}(0, 1)'$. The above characterization then reads

$$\lambda(x) = \lim_{t \pm\infty} \frac{1}{t} \log \max\{|\mu_1^t x_1|, |\mu_2^t x_2|\} = \lambda_1 \Leftrightarrow x \in L(\lambda_1) = \mathrm{span}(1, 0)'.$$

Thus, the if-and-only-if statement only holds for the two-sided limit because for $t \to -\infty$ the above expression is dominated by $|\mu_2|^t$. The same argument applies for λ_2 and $L(\lambda_2)$.

In order to prepare for the random coefficient case, it is instructive to invoke the *spectral theorem* (Meyer, 2000, chapters 7.2 and 7.3). let A be diagonalizable with spectrum $\sigma(A) = \{\mu_1, \ldots, \mu_\ell\}$, then

$$A^t x = \mu_1^t \pi_1 x + \cdots + \mu_\ell^t \pi_\ell x$$

where π_j denote the projectors onto $\mathbf{N}(A - \mu_j I_d)$ along $\mathbf{R}(A - \mu_j I_d)$, $j = 1, \ldots, \ell$. They have the properties that $\pi_i \pi_j = 0$ whenever $i \neq j$ and that $\pi_1 + \ldots \pi_\ell = I_d$. Hence, for an arbitrary $x \in \mathbb{R}^d$ the asymptotic behavior is dominated by the largest eigenvalue in absolute terms, say μ_1. If, however, x is chosen specifically as an element of the complement of $\mathbf{N}(A - \mu_1 I_d)$, i.e. such that $\pi_1 x = 0$, the asymptotic behavior is governed by the second largest eigenvalue in absolute terms. Obviously, one proceed further in this manner until all eigenvalues are exhausted. From the diagonal representation of A, the projections π_j are given by

$$\pi_j = Q \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{d_j} & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1}, \qquad j = 1, \ldots, \ell$$

where d_j is the dimension of the eigenspace related to eigenvalue μ_j and where the columns of Q consist of generalized eigenvectors which form a basis of $\mathbb{R}^d$.

B More on the MET

Relation to Birkhoff's ergodic theorem To justify the labeling *ergodic* in the MET, I relate it to Birkhoff's ergodic theorem. Let the state space be one-dimensional, i.e. $d = 1$. Then define $f(\omega) = \log |a(\omega)|$ where $a(\omega)$ stands for $A(\omega)$. The ergodicity of θ and the integrability condition imply that the prerequisites of Birkhoff's pointwise ergodic theorem are satisfied.[17] Thus,

$$\frac{1}{t} \sum_{j=0}^{t-1} f(\theta^j \omega) \longrightarrow \int_\Omega f \mathrm{d}\mathbf{P} = \lambda.$$

As $\log |\varphi(t, \omega, x)| = \sum_{j=0}^{t-1} \log |a(\theta^j \omega)| + \log |x|$, statement *(iii)* in the MET corresponds exactly the Birkoff's theorem because $\lim_{t \to \pm\infty} \frac{1}{t} |x|$ converges to zero. Birkhoff's theorem, however, cannot be immediately generalized to higher dimensions because the matrix multiplication is not commutative.

The case of triangular matrices To get a more profound understanding of the MET, it is instructive to examine the case of triangular matrices.[18]

[17] An introduction to ergodic theory can be found, f.e., Silva (2008, chapter 5), Grimmett and Stirzaker (2001, section 9.5), or Colonius and Kliemann (2014, section 10.1).

[18] This exposition follows Arnold (2003, p.129–130) and Berger (1993, p.155).

Let $A(\omega) \in \mathbb{GL}(2)$ be upper triangular:

$$A(\omega) = \begin{pmatrix} a(\omega) & c(\omega) \\ 0 & b(\omega) \end{pmatrix}, \qquad a(\omega) \neq 0 \text{ and } b(\omega) \neq 0.$$

$\Phi(t)$ is then given by

$$\Phi(t) = A_{t-1} \dots A_1 A_0 = \begin{pmatrix} \prod_{j=0}^{t-1} a_j & \sum_{k=0}^{t-1} a_{t-1} \dots a_{k+1}\, c_k\, b_{k-1} \dots b_0 \\ 0 & \prod_{j=0}^{t-1} b_j \end{pmatrix}.$$

Note that $\mathbb{R}e_1 = \operatorname{span}(1,0)'$ is an invariant subspace for the $\Phi(t)$'s. Let the stochastic sequences $\{a_t\}$, $\{b_t\}$, and $\{c_t\}$ be ergodic with $\alpha = \mathbb{E}\log|a_t|$, $\beta = \mathbb{E}\log|b_t|$, and $\gamma = \mathbb{E}\log|c_t|$. Then

$$\frac{1}{t}\sum_{j=0}^{\infty} \log|a_t| \to \alpha \quad \text{ and } \quad \frac{1}{t}\sum_{j=0}^{\infty} \log|b_t| \to \beta.$$

Hence,

$$\frac{1}{t}\log|\det\Phi(t)| \to \alpha + \beta.$$

From implication *(v)* of the MET, it follows that $\lambda_1 + \lambda_2 = \alpha + \beta$. Obviously, the Lyapunov exponents of $[\Phi(t,\omega)]_{11}$ and $[\Phi(t,\omega)]_{22}$ are α and β, respectively. Moreover, the Lyapunov exponent of $[\Phi(t,\omega)]_{12}$ is less than or equal to $\max\{\alpha,\beta\}$. The subadditivity of the lim sup implies $\lambda(x+y) \leq \max\{\lambda(x),\lambda(y)\}$, with equality if $\lambda(x) \neq \lambda(y)$, hence using the Euclidean norm in $\mathbb{GL}(2)$

$$\frac{1}{t}\log\|\Phi(t,\omega)\| \to \lambda_1 = \max\{\alpha,\beta\}.$$

Thus,

$$\lambda_1 = \max\{\alpha,\beta\} > \frac{\alpha+\beta}{2} > \lambda_2 = \min\{\alpha,\beta\} \quad \text{for } \alpha \neq \beta.$$

When $\alpha = \beta$, $\lambda_1 = \alpha = \beta$ with multiplicity 2.

To compute the Lyapunov spaces, assume without loss of generality $\alpha > \beta$, hence $\lambda_1 = \alpha$ and $\lambda_2 = \beta$. For any vector $x = (x_1, 1)'$ to grow at rate β, x must be an eigenvector with respect to eigenvalue $b(t) = [\Phi(t)]_{22}$. Thus

$$\begin{pmatrix} a(t) & c(t) \\ 0 & b(t) \end{pmatrix} \begin{pmatrix} x_1 \\ 1 \end{pmatrix} = b(t) \begin{pmatrix} x_1 \\ 1 \end{pmatrix}$$

with $a(t) = [\Phi(t)]_{11}$ and $c(t) = [\Phi(t)]_{12}$. Taking limits and recognizing that $b(t)/a(t) \to 0$, one obtains

$$x_1 = -\lim_{t\to\infty} \frac{c(t)}{a(t)} = -\sum_{k=0}^{\infty} \frac{c_k b_{k-1} \dots b_1 b_0}{a_k \dots a_1 a_0}.$$

This defines the random Lyapunov subspace $L(\lambda_2) = L(\beta) = \mathrm{span}(x_1, 1)'$. Because x has to grow at rate β, x must be random. Moreover, this randomness depends on the entire sequence $\{A_t\}$. The other Lyapunov subspace is, as noted before, $L(\lambda_1) = L(\alpha) = \mathbb{R}e_1$.

C Proof of Theorem 2

The first part of the proof replicates the proof proposed by Arnold and Crauel (1992) and Arnold (2003, theorem 5.6.5) omitting some of technical details.

Proof. First I prove that the expressions for $x_t^{(b)}$ and $x_t^{(f)}$ are well-defined. Consider for this purpose a trajectory starting in x denoted by $\varphi(t, \omega, x)$. Iterating the difference equation backward

$$\varphi(t, \omega, x) = \Phi(t, \omega)x + \Phi(t, \omega) \sum_{j=0}^{t-1} \Phi(t-j, \omega)^{-1} b(\theta^{t-1-j}\omega).$$

This implies

$$\begin{aligned}
\pi^s(\theta^t\omega)\varphi(t, \omega, x) &= \pi^s(\theta^t\omega)\Phi(t, \omega)x \\
&\quad + \pi^s(\theta^t\omega)\Phi(t, \omega) \sum_{j=0}^{t-1} \Phi(t-j, \omega)^{-1} b(\theta^{t-1-j}\omega) \\
&= \Phi(t, \omega)\pi^s(\omega)x \\
&\quad + \Phi(t, \omega) \sum_{j=0}^{t-1} \Phi(t-j, \omega)^{-1} \pi^s(\theta^{t-j}\omega) b(\theta^{t-1-j}\omega) \qquad \text{(C.1)}
\end{aligned}$$

Next choose $\beta \in (0, \min\{-\lambda_s - \kappa, \lambda_u + \kappa\}$ where κ satisfies $\lambda_s = \max_{\lambda_j<0} \lambda_j < -\kappa < 0 < \kappa < \lambda_u = \min_{\lambda_j>0} \lambda_j$. The first term in the above expression then converges exponentially fast to zero because $\left\| \Phi(t, \omega)|_{L^s(\omega)} \right\| \leq e^{-\beta t}$.

The integrability condition $\mathbb{E} \log^+ \|b\| < \infty$ implies

$$\sum_{j=0}^{\infty} \mathbf{P} \left[\log^+ \|b(\theta^{t-j-1}\omega)\| > \frac{\beta}{2} \right] < \infty.$$

This implies by the Borel-Cantelli lemma that almost surely

$$\limsup_{j\geq 0} \frac{1}{j} \log^+ \|b(\theta^{t-j-1}\omega)\| \leq \frac{\beta}{2}.$$

Hence

$$\limsup_{j\geq 0} \frac{1}{j} \log \|\Phi(j, \theta^{t-j}\omega)\pi^s(\theta^{t-j}\omega)b(\theta^{t-1-j}\omega)\| \\ \leq \limsup_{j\geq 0} \frac{1}{j} \log \|\Phi(j, \theta^{t-j}\omega)\| \; \|\pi^s(\theta^{t-j}\omega)b(\theta^{t-1-j}\omega)\| \leq \frac{-\beta}{2}.$$

The second term in equation (C.1) therefore is

$$\Phi(t,\omega) \sum_{j=0}^{t-1} \Phi(t-j,\omega)^{-1}\pi^s(\theta^{t-j}\omega)b(\theta^{t-1-j}\omega) \\ = \sum_{j=0}^{t-1} \Phi(j, \theta^{t-j}\omega)^{-1}\pi^s(\theta^{t-j}\omega)b(\theta^{t-1-j}\omega)$$

converges almost surely to $x_t^{(b)}$. Thus, $x_t^{(b)}$ is well-defined. A similar argument can be made with respect $x_t^{(f)}$

Using the towering property of conditional expectations and omitting the dependence on ω whenever possible, the solution given in equation (3.1) indeed solves the expectational difference equation:

$$\begin{aligned} \mathbb{E}_t x_{t+1} = \mathbb{E}_t \Bigg\{ & \Phi(t+1)x + \Phi(t+1) \sum_{j=0}^{\infty} \Phi(t+1-j)^{-1}\pi^s(\theta^{t+1-j}\omega)b_{t-j} \\ & - \Phi(t+1)\mathbb{E}_{t+1}\left[\sum_{j=0}^{\infty} \Phi(t+j+2)^{-1}\pi^u(\theta^{t+j+2}\omega)b_{t+1+j}\right]\Bigg\} \\ = {} & A_t\Phi(t)x + \Phi(t+1)\Phi(t+1)^{-1}\pi^s(\theta^{t+1}\omega)b_t \\ & + A_t\Phi(t) \sum_{j=0}^{\infty} \Phi(t-j)^{-1}\pi^s(\theta^{t-j}\omega)b_{t-1-j} \\ & - A_t\Phi(t)\mathbb{E}_t\left[\sum_{j=0}^{\infty} \Phi(t+j+1)\pi^u(\theta^{t+j+1}\omega)b_{t+j}\right] \\ & + \Phi(t+1)\Phi(t+1)^{-1}\pi^u(\theta^{t+1}\omega)b_t \\ = {} & A_t x_t + (\pi^s(\theta^{t+1}\omega) + \pi^u(\theta^{t+1}\omega))b_t = A_t x_t + b_t. \end{aligned}$$

The solution (3.1) depends parametrically on the unknown $x(\omega)$. In order to fix $x(\omega)$, I resort to the initial value condition $Rx_0(\omega) = c$ which implies $c - Rx_0^{(p)}(\omega) = Rx(\omega)$. The boundedness condition further requires that $x(\omega) \in L^s(\omega)$ or, equivalently, that $\pi^u(\omega)x(\omega) = 0$. These two conditions then determine $x(\omega)$ uniquely if the rank condition (3.2) is satisfied. □

D Computing Lyapunov Exponents

Although the Lyapunov exponents and the corresponding subspaces are defined in a straightforward manner on the theoretical level, it is not a straightforward task to compute them numerically. The reason for this difficulty stems from the exponential growth of the elements in $\Phi(t)$, respectively Υ, as t becomes large. Trying to compute these matrices directly very quickly hits the numerical bounds of any computer. To avoid this problem iterative QR and SVD decompositions have been proposed (see Dieci and Elia, 2008).

In this paper, I use the QR approach which is very easy to implement. The algorithm is initialized by taking some $X(0) = A_0$ as a starting value. Let the QR decomposition of $X(0)$ be given as $X(0) = Q_0R_0$ where Q_0 is an orthogonal matrix and R_0 an upper triangular matrix.[19] Then compute $X(1) = A_1X(0)$ and perform the QR decomposition of $X(1)Q_0 = Q_1R_1$. Obviously, $X(1)X(0) = Q_1R_1R_0$. Proceeding in this way, the QR decomposition of $\Phi(t) = X(t-1)\ldots X(1)X(0)$ is obtained:

$$\Phi(t) = Q_{t-1}R_{t-1}\ldots R_1R_0.$$

Generalizing the arguments made in Section 2.3 for triangular 2×2 matrices to $d \times d$ matrices, one gets

$$\lambda_j = \lim_{t\to\infty}\frac{1}{t}\log\prod_{k=0}^{t-1}[R_k]_{jj} = \lim_{t\to\infty}\frac{1}{t}\sum_{k=0}^{t-1}\log\,[R_k]_{jj},\quad j = 1,\ldots\ell,$$

where $[R_k]_{jj}$ is the j-th diagonal element of R_k. The algorithm stops when a sufficient precision is obtained. For further details see Dieci and Elia (2008) and the literature cited therein.

[19] Given the assumption of the invertibility of the matrices A_t, the QR decomposition is unique if the diagonal elements of R are taken to be strictly positive.